Simulation modeling and optimization of competitive electricity markets and stochastic fluid systems

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geboren op 30 juli 1979 te İzmir, Turkije.

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Özge Özdemir
Amsterdam, January 2013

Contents

Chapter 1

Introduction

1.1 Scope of the Thesis

The research in this thesis deals with utilizing operations research and simulation techniques to model and solve some interesting stochastic problems which are complex in nature. Two of these problems are applications in energy and economics in relation to modeling generation capacity investments in deregulated electricity markets, which have gained a lot of attention since the restructuring of electricity markets towards decentralization. Another particular group of problems is highly relevant for manufacturing and inventory control in relation to applications in stochastic fluid systems.

One most commonly used modeling approach in this thesis is the utilization of *optimization problems*. Development of efficient solution algorithms and high speed computers has made optimization problems (e.g., linear and nonlinear programs) attractive and widely-used to solve various decision making problems. An optimization problem is generally a problem of one decision maker. One may also be interested in solving optimization problems involving two or more decision makers simultaneously. We then speak of an *equilibrium problem* and the solution of the corresponding problem is called an *equilibrium*. The concept of equilibrium, as introduced by John Forbes Nash (Nash (1950)) in noncooperative game theory, plays a central role in economics. Operations Research theory and techniques have also been extensively utilized for analyzing and solving various types of equilibrium problems in the literature (e.g., Karamadian (1971), Harker and Pang (1990), Harker (1993), Ferris and Pang (1997), Ferris and Kanzow (2002), Facchinei and Pang (2003)). Many problems in engineering and economics can be formulated as equilibrium problems. An excellent overview of how this is accomplished is given by Ferris and Pang (1997).

One of the most interesting and recent applications of equilibrium problems in economics include electricity markets which have experienced significant changes towards deregulation and competition with the aim of improving economic efficiency. In this thesis, we mainly focus on the modeling of generation capacity investments in deregulated electricity markets. We utilize the theory of equilibrium and equilibrium problems to model the behavior of both competitive and strategic generators in electricity markets. Utilizing theory and applications of equilibrium problems in electricity markets, the scope of this thesis also falls in the intersection of operations research and noncooperative game theory.

Under the old-style vertically integrated monopoly structure of electricity markets before deregulation, prices were regulated and set to cover total costs and hence enable the utility to finance investment. In the ideal situation of a well-managed regime, this corresponds to utilizing a single optimization problem (which fitted well with the regulated monopoly regime with a central planner) to identify optimal planning and operation of electricity systems. Since deregulation, the power generation sector has operated in a competitive environment where electricity prices are set by an unregulated wholesale market. Similarly, investments in power generation is no longer done by a central planner and are left to market participants responding to current and expected market prices. Thus, long term planning of power generation capacity may suffer from uncertainty and imperfections in the short-term (e.g., day ahead and real time) operation of electricity markets. This has raised the concern about incentives to invest in generation capacity in the literature as well as among policy makers since deregulation of electricity markets. Many authors argue that low and volatile spot market prices in current energy-only markets, reflecting short-run marginal costs and constrained by low price caps, may not provide adequate incentives for sufficient generation capacity investments (e.g., Stoft (2002), Cramton and Stoft (2005), Joskow (2007), Joskow (2008)).

Additional market mechanisms may be designed by regulators to eliminate the imperfections which reduce the incentives in investments. We assess three examples of these market mechanisms under perfect competition: an energy-only electricity market with VOLL pricing (reflecting the value of lost load (VOLL), see Stoft (2002)), an electricity market with an additional forward capacity market (e.g., Cramton and Stoft (2005), Hobbs et al. (2007), Joskow (2008)), and an electricity market with operating-reserve pricing (e.g., Stoft (2002), Hogan (2005), Hogan (2009)). We elaborate on both how these market mechanisms can be represented by mathematical models and the assessment of equilibrium for generation capacity investments under different market designs when future spot market conditions are not known in advance (i.e., under demand uncertainty). Under each market design, the investment decisions of generators are formulated as a two-stage equilibrium problem where generation capacities are installed in the first stage and generation takes place in future spot market at the second stage. We emphasize the link between optimization and equilibrium problems and show that, regarding the problem of generation capacity investments in perfectly competitive electricity markets, the equilibrium problems of most of these market mechanism can be cast as optimization problems (e.g., (non)linear programs, two-stage stochastic programs). In particular, this result indicates the prevalence of optimization problems, which are fast and efficient to solve, for providing solutions to stochastic equilibrium models of real world systems with uncertainty. This result also suggests that an equilibrium of a single-stage model (so called *open loop model*) in which investment and operation decisions are made simultaneously coincides with an equilibrium of a two-stage model (so called *closed loop model*) where investment and operation decisions are made sequentially. In one case of these market mechanisms, namely operating reserve pricing, we cannot cast the corresponding equilibrium problem as an optimization problem.

We also consider a closed loop model of strategic generators facing uncertain demand and anticipating perfectly competitive spot market outcomes when they choose their generation capacities, in which case an equilibrium of the closed loop model does not coincide with an equilibrium of the less complicated open loop model as shown by Wogrin et al. (2012). In general, such closed loop models can be formulated as an equilibrium problem with equilibrium constraints (EPEC) and examples have been posed in the literature that have multiple or no equilibria (e.g., Hu (2003), Ehrenmann (2004), Hu and Ralph (2007), Gabriel et al. (2012)). This indicates that an equilibrium for the corresponding two-stage game may not exist or, if exists, it is in general not unique. Therefore, it is of interest for users of these models to know general sets of conditions under which solutions exist and are unique so that they can have confidence in the solutions before solving such complicated closed loop models. Thus, establishing general sets of conditions for existence and uniqueness enhances the value of such models for policy and market intelligence purposes. The existence and uniqueness of the closed loop equilibrium in electricity markets has been extensively discussed in the literature under various assumptions (e.g., competition in quantities or prices, price-demand relations, number of firms in the market, symmetric or asymmetric costs of firms etc.). Among these, Kreps and Scheinkman (1983), Gabszewicz and Poddar (1997), Murphy and Smeers (2005), and Grimm and Zoettl (2008) focus on the equilibrium of generation capacity investments in the context of addressing long term aspects of market power. In this thesis, we also establish sufficient conditions which guarantee the existence and uniqueness of the closed loop equilibrium for the generation capacity investments of strategic generators in electricity markets. Our analysis is done under a broader scope in terms of the underlying price-demand relations, the assumed demand uncertainty with a broad class of continuous distributions, and any finite number of generators with symmetric or asymmetric costs.

Despite the impressive advances in mathematical modeling, some complex systems are well beyond the capabilities of being represented with explicit mathematical expressions. Even when it is plausible to formulate a proper mathematical model, the resulting optimization problem may prove too complex for available solution algorithms. An alternative approach to modeling complex systems is *simulation*. In simulation models, the relation between the input (e.g., parameter) and output (e.g., performance measure) need not to be stated explicitly. Instead, the system is represented by components such as *entities* (e.g., object of interest), *events*, and *states*, and the actual behavior of the system is imitated by mathematical or logical relationships to link these components. Given a set of input parameters, a simulation model is run and the output is realized.

In this thesis, we also utilize a simulation modeling approach, the so called *generalized semi-Markov Process (GSMP)*, to model a manufacturing flow control problem proposed by Kimemia and Gershwin (1983) within the framework of optimal control problems for stochastic fluid systems. Glynn (1989) provides a brief introduction on the role of GSMPs in simulation and an excellent description of GSMP framework can be found in Shedler (1993). Although in the past GSMPs were mainly used for modeling systems with discrete entities, they can also be well suited for modeling fluid systems; see Suri and Fu (1994) and Gürkan (2000) for example. The particular manufacturing flow control problem that we consider involves optimization of multiple target inventory levels in a production/inventory system. In this system, the production and demand rates change according to some stochastic process. There has been a strong interest in the development of production and inventory control models and in their analysis within this framework. Utilization of GSMP enables us to develop an intuitive and transparent algorithm to obtain a flexible and general representation of the corresponding problem.

Decision making problems with fixed or known parameters are called *deterministic* problems. Decision making problems under uncertainty, where one or more parameters vary or are unknown at the time a decision should be made, are called *stochastic* problems. Many real world decision making problems in engineering and economics give rise to intractable mathematical formulations, especially due to their stochastic nature. A key difficulty in decision making under uncertainty is in dealing with an uncertainty space that is huge and frequently leads to very large-scale models. Additional complexity is difficulty in derivation of closed-form analytic expressions for the performance measures in many stochastic problems. Alternative methods to find a solution to stochastic problems is an active area of research. Among these, we focus on a class of simulation-based methods, so called *sample-path methods* (also known as *sample average approximation*) (e.g., Robinson (1996) and Rubinstein and Shapiro (1993)). Regarding sample-path methods, the complex nature of the stochastic systems motivates researchers to use simulation together with deterministic solution methods as an alternative tool for solving stochastic optimization problems or stochastic equilibrium models. There has been a lot of progress in the development and implementation of sample-path methods for optimizing the performance of a complex stochastic system that can be observed only through simulation (where no analytic solutions available) (e.g., Plambeck et al. (1996), Shapiro and Homem-De-Mello (1998), Gürkan et al. (1999b), Gürkan (2000)). In many applications, the performance function to be observed is a limit function which is the output of a sequence of simulation runs of increasing length, all using the same random number streams; that is if we go out far enough in the sample-path, we get a good estimate of the limit function. The resulting limit function is deterministic such as an expectation or a steady-state performance measure from a simulation model. The optimization or equilibrium problem with the resulting deterministic limit function can be solved by using state-of-the-art deterministic optimization and equilibrium solvers, respectively. Most of the state-of-the-art solvers require gradient information; therefore we also exploit the developments in gradient estimation techniques to calculate the exact partial derivatives. Gürkan et al. (1996, 1999a) give an extension of sample-path methods to find an approximate equilibrium of stochastic equilibrium models. The basic idea again is utilization of simulation and efficient and effective deterministic solvers (together with gradient estimation techniques) for solving complex stochastic equilibrium problems.

To summarize, this thesis mainly focuses on

(i) the application of deterministic or stochastic optimization and equilibrium problems to model and analyze the problem of generation capacity investments by competitive and strategic generators in decentralized electricity markets, which has been a research topic of interest since the deregulation of electricity markets,

(ii) the application of simulation and stochastic fluid models to model and analyze a stochastic manufacturing flow control problem, and

(iii) the application of sample-path methods together with the state-of-the-art deterministic optimization tools to be able to solve these problems which are complex and large scale due to their stochastic nature.

1.2 Research Topics and Contribution

In this section, we give an overview of our research topics and contribution, which are later discussed in detail in the succeeding chapters.

1.2.1 Electricity Markets: From Monopoly to Competition

For decades, electricity markets all over the world were particularly organized as public monopolies. Since the late 1980s, policy makers and regulators in many countries have liberalized, restructured or deregulated the electric power sector, typically by introducing competition at the generation and retail level. Reforms were pioneered by Chile, the UK, and Norway and have spread all over the world. The motivations for changing the power industry structure and the implementation of market reforms vary from country to country, but in general the aim of restructuring has been to increase efficiency and competition leading to lower costs and price reductions. Today different paradigms of restructured electricity systems, varying in complexity and scope, can be found in many parts of the world including United States, New Zealand, Australia, and Europe.

The supply of electricity from generators to consumers requires three basic steps: generation, transmission, and distribution. Generation takes place at power stations where energy fuel (e.g., gas, coal, oil) is converted to electricity. Electricity is then transmitted over long distances from power stations to substations by Transmission System Operators (TSOs). Transmission takes place via high voltage lines in order to minimize losses. Electricity is transformed to a lower voltage at the substations and distributed to consumers through low voltage lines by Distribution System Operators (DSOs). The international experience from electricity liberalisation around the world has led to consensus on some steps of restructuring for achieving a well functioning electricity market (see Sioshansi and Pfaffenberger (2006) and Sioshansi (2008) for a survey). In the process of restructuring, the former vertically integrated electricity utilities are restructured and unbundled, and competition has been introduced into generation, wholesale, and retail segments of the industry. Transmission and distribution businesses usually remain as national or regional monopolies but they are regulated by an independent system operator. The effective separation of generation and transmission activities is crucial for achieving competition in electricity markets. As a result, electricity generators owning one or different types of power stations compete to supply power for consumers and maximize their profits. Generators also need to have access to transmission and distribution services provided by TSOs and DSOs in order to reach their customers. Other common elements of the restructuring include introduction of wholesale and spot power markets and establishment of independent system operator.

Since the introduction of competition in electricity markets by decentralization of electricity generation, many interesting research questions have been addressed in the intersection of operations research and noncooperative game theory in economics. Most of these questions originate from the challenges faced by regulators and policy makers due to the specific characteristics of electricity markets. In general, an electricity market shows different characteristics than other economic wholesale markets, leading to complex interactions between the market participants (Borenstein and Bushnell (1999), Stoft (2002)). First, electricity is extremely costly to store. This implies that generated electricity must be transmitted and consumed directly at every second. A shortfall or surplus of electricity can endanger the stability of the entire electricity network due to the complex nature of the way electricity flows through the network and because of the need to protect network equipments from large variances in frequency and voltage. If the demand for power exceeds the supply, this may lead to the action of shutting down generation plants and transmission equipment which, in the worst cases, can lead to a major regional blackouts. Another difficulty is that demand tends to be inelastic, meaning that consumers cannot respond to real time prices. In order to respond to real time prices, final consumers would need to have real time meters. This is not currently the case for a large proportion of consumers. Hence, when there is shortage of supply, transmission operator has to curtail consumption at very high costs. The combination of excluding demand-side response with the complexities in nature of the market (costly storage and grid reliability requirements) makes electricity markets vulnerable to the exercise of market power. Most of these complexities can be overcome by sufficiently well designed markets. Thus, extension of optimization models of a single decision maker to equilibrium models of multiple decision makers has become an essential instrument in the literature for analyzing different market designs for decentralized electricity markets (e.g., Wolf and Smeers (1997), Boucher and Smeers (2001), Hobbs (2001), Day et al. (2002), Metzler et al. (2003), Hobbs and Pang (2004), Schulkin et al. (2010)).

1.2.1.1 Generation Capacity Investments in Perfectly Competitive Electricity Markets

In the days of regulated monopolies, there was no issue of generation resource adequacy. Companies were obligated to serve the demand and had to invest accordingly; an optimization model was used to compute the expansion of generation capacity decisions that would satisfy demand at minimal investment and operations costs. In compensation for this obligation, electricity prices were regulated (often at average cost) in a way that guaranteed that the company could pay for its expenses (including reimbursement of long term debt) and make a reasonable profit on equity. Capacity expansion models were extensively developed during the regulatory periods before loosing some of their appeal after restructuring. The theory of peak load pricing for non-storable commodities, such as electricity, is the economic counterpart of these computational models; it can be seen as an economic interpretation of the capacity expansion model in terms of electricity prices that induce efficient investments and operations. Of particular importance, the theory of peak load pricing explains that the price of electricity in the highest demand period must embed a particular (peak load) component to induce an efficient capacity mix. Both the capacity expansion models and the theory of peak load pricing can be traced back to work conducted in Électricité de France in the fifties and sixties (see the collection of early papers treating both subjects in Morlat and Bessière (1971)).

With liberalisation, generation and investments became the responsibility of companies who had to make a profit on the electricity market. This gave rise to the question whether these markets would provide sufficient incentives for investments or whether additional policy measures would be needed to ensure these investments and thereby security of delivery in electricity markets. This discussion focuses especially on those power plants which will only be needed to meet demand at peak hours and therefore have to earn sufficient revenues in those hours to cover their investment costs. The theory of peak load pricing, which was initially developed for the regulated monopoly, was later proved equally relevant to perfectly competitive markets (see Crew et al. (1995) for a survey). This theory has become crucial today to explain why perfectly competitive energy-only electricity markets, which are based on power exchanges where supply bidding is at the sole short-run marginal cost, may not spontaneously provide the right incentive to invest in generation capacity for peak load. To clarify further, in a perfectly competitive energy-only spot market (lacking demand-side response), the market price is set at the bid of the most expensive operating power plant (e.g., peak power plant). Thus, when there is sufficient capacity, market price is equal to the fuel cost of the peak power plant, leaving it with a zero margin to cover its investment costs. Furthermore, unacceptability of price spikes and concerns on market power have led the politicians to cap the prices during scarcity hours (i.e., when generation capacity is scarce) and potentially limit the peak generators from receiving sufficient margin to cover their investment cost. This missing margin is now commonly referred to as the missing money in the literature (e.g., Stoft (2002), Cramton and Stoft (2005, 2006) and Joskow (2007, 2008)).

Since the resource adequacy problem due to missing money was brought up, generation capacity investments in decentralized electricity markets have gained a lot of attention as a research topic and in the political debate with an emphasis on the questions: Do current liberalized markets (e.g., energy-only markets) provide right incentives to invest in generation capacity? If not, what type of market mechanisms are needed to provide the right incentives?

Initially, when liberalisation was introduced in Europe and in the US, policy makers held the view that energy-only electricity markets would provide sufficient incentives for new investment. Therefore, no additional measures were put in place to stimulate investments. However, experience with these energy-only markets have led policy makers to change their views. For example, in recent years, more and more US states have implemented some sort of policy measures aimed at stimulating investments in new generation capacity, such as scarcity pricing when generation capacity is inadequate or capacity payments to generators via additional capacity markets next to the electricity market

The concern of generation adequacy have also been raised recently in Europe; in particular as a result of the substantial growth of intermittent renewables. The generation from intermittent renewables such as wind is highly volatile and flexible conventional generation capacity (e.g., gas-based power plants) is needed as a back-up for limited number of hours in a year when wind is not available. In an energy-only electricity market, renewables (e.g., wind, solar) would increase price volatility, tend to reduce market price levels in most of the hours, and decrease the overall capacity utilization of conventional capacity (e.g., gas, coal). This may aggravate the missing money problem in particular for conventional resources and investments in conventional capacity becomes less attractive. Therefore, many market participants and policymakers in Europe are now expressing a concern whether energy-only markets with high penetrations of intermittent renewables are capable of providing sufficient incentives for back-up generation capacity in future. For example, the debate in Northwest Europe concerning the need for investments in electricity generation capacity is intense for some countries such as Germany and the UK. The UK has significant needs in investments as of 2015 (De Joode et al. (2013)). These investment are needed mainly as a replacement of old capacity that has to be decommissioned (such as coal plants) partly due to environmental regulations. One of the reasons for the large investments needed in Germany is the planned nuclear phase-out. Furthermore, back-up capacity is needed to accommodate the increasing share of renewables (Elberg et al. (2012)). Elberg et al. (2012) investigate the missing money problem in Germany with high penetration of intermittent renewables after 2020. They confirm that gas-based generation units may not be able to recover their investment costs in a future energy-only electricity market with a higher level of intermittent renewable energy sources. Their study underlines that the share of intermittent renewable electricity generation itself is not at the root of the problem but can add to the challenges of providing long-term resource adequacy in an energy-only electricity market. Similar findings are also confirmed by Özdemir et al. (2013) for the Netherlands with higher level of intermittent renewables in 2020.

Parallel to the policy debate, there has also been considerable attention in the scientific literature for the question whether investors can recoup their generation capacity investments on the electricity market. According to this literature, an optimal energy-only electricity market would allow investors in peak capacity to earn sufficient revenues during peak hours (e.g., Stoft (2002), Joskow (2008)). Important conditions for such an optimal energy-only market are prices which are allowed to rise to the point where consumers would prefer to be shut off instead of paying this price and sufficient flexibility in demand to react to high prices. However, these conditions are not necessarily met in real energy-only electricity markets. For instance, high peak prices tend not be acceptable for politicians and regulators, who therefore have introduced price caps and most of the electricity consumers cannot respond to prices. Such imperfections in addition to uncertainties in energy-only markets may lead to insufficient net revenues for generators to provide adequate generation capacity for optimal generation capacity portfolio. Theoretical and numerical examples are given in the literature to illustrate a variety of imperfections in energy-only markets leading to this missing money problem and improvements are suggested to reduce its magnitude (e.g., Hobbs et al. (2001), Stoft (2002), De Vries (2004), Cramton and Stoft (2005), Hogan (2005), Hobbs et al. (2007), Joskow (2008), Cramton and Ockenfels (2011)). Among these, we focus on remedies which improve (scarcity) pricing when generation capacity is fully utilized.

In Chapter 3, we assess three different market designs that have been discussed in the literature to remedy resource adequacy problem: an energy-only electricity market with VOLL pricing (a variant of scarcity pricing reflecting the value of lost load (VOLL)), an electricity market with an additional forward capacity market, and an electricity market with operating-reserve pricing. In an energy-only electricity system, when there is shortage of supply, the price of electricity should be sufficiently high to provide incentives for adequate generation capacity investments, known as VOLL pricing. In perfectly competitive markets, Stoft (2002) shows that VOLL pricing results in optimal generation capacity investments. However, in reality VOLL is difficult to estimate and therefore a price cap (which is in general lower than VOLL) is used. If the price cap induced by the regulator is not high enough, this will constrain prices from rising up to their competitive levels at peak hours, yielding underinvestment in generation capacity (Joskow (2008)). An alternative solution is to implement a capacity market where the regulator imposes some capacity target in line with expected demand and the firms contributing to the sufficient investment level receive capacity payments accordingly. In order to increase the robustness against market power, many variants of capacity markets have been proposed regarding the implementation of the market design and the treatment of demand response (e.g., Cramton and Stoft (2005), Joskow (2008), Hobbs et al. (2007), Cramton and Ockenfels (2011)). In this thesis, we assume a forward capacity market where capacity is auctioned before the investment decision is made and the resulting capacity payment is certain for the lifetime of the plant. A more sophisticated alternative remedy is to apply some form of reliability or operating-reserve pricing so that electricity price increases when the reserve margin decreases (Stoft (2002), Hogan (2005), Hogan (2009)). Lastly, as a categorically different alternative, ensuring a price responsive demand in the short-run as well as in the long-run may remedy the resource adequacy issue. Next, we discuss our analysis and contributions.

We concentrate on these four mechanisms (including demand-side bidding) and propose alternative ways to model and solve the generation capacity investment problem of multiple firms in a competitive environment. We also take into account uncertainties in future and extend the deterministic formulations of the electricity market models with stochastic elements (e.g., demand uncertainty) which may lead to large scale problems. The natural approach is to resort to complementarity formulations as this mathematical programming paradigm has been extensively used to model restructured electricity systems. We assess the extent to which these models can be restated as stochastic optimization problems and propose solution algorithms using sample-path methods. Consequently, we can use deterministic optimization solvers, that are numerous and quite powerful, to solve stochastic equilibrium problems. This result is of interest to users of such models for policy analysis purposes since practical tools are needed in real world problems to gain insights into the social implications of a market design or a policy target (see e.g., Schroeder (2012) and Allcott (2012) for examples of real world applications).

We consider perfectly competitive electricity markets of three types of agents: risk neutral generators, a transmission system operator, and consumers. As mentioned above, perfectly competitive energy-only electricity markets (barring demand-side bidding or scarcity pricing) may not guarantee resource adequacy since the market sets the price at the bid of the most expensive operating plant (peak power plant). Thus, even in perfectly competitive markets, guaranteeing resource adequacy may require market designs creating investment incentives by provision of enough margin to cover the capital costs of an efficient generation system, including the peak power plant. For each market design, we model and analyze capacity investment choices of each firm using a two-stage optimization problem where its generation capacities are installed in the first stage and its generation takes place in future electricity market at the second stage. Since there are multiple firms, we essentially have an equilibrium problem at both stages.

In reality, future electricity market conditions are not known in advance. Although we focus mainly on electricity demand being uncertain and/or fluctuating with a very low elasticity, our methodology and results can be easily generalized to spot markets with other uncertainties (i.e., fuel costs, transmission capacities, wind generation etc.). For given wind capacity and volatile wind generation, one can substitute demand with residual demand levels (i.e., demand minus wind). Under uncertainty about future electricity market conditions, real world problems including generation capacity investment decisions lead to stochastic equilibrium problems that are large scale and computationally more complex to solve than a stochastic optimization problem; see e.g., Schroeder (2012) and Allcott (2012) for examples of real world applications. Equilibrium problems are indeed broader than optimization problems which is addressed in detail by Gabriel et al. (2012). In this thesis, we not only emphasize the link between optimization and equilibrium problems but also show that, regarding the problem of generation capacity investments in perfectly competitive electricity markets, most of the formulations of stochastic equilibrium problems can be cast as two-stage stochastic programs. The main contributions of Chapter 3 can be summarized as follows:

- We expand the existing short-run models of restructured electricity system to include both uncertainty of spot market conditions (i.e., demand uncertainty) and resource adequacy mechanisms, such as VOLL pricing, capacity market, and operating-reserve pricing, and also demand-side bidding as an alternative remedy.

- In perfect competition, we show that most of the formulations of these "(stochastic) equilibrium" models are in fact equivalent[1] to or can be cast as single (stochastic) nonlinear or linear optimization problems. In particular, in the stochastic setting, this result indicates the prevalence of two-stage stochastic programming for providing solutions to stochastic equilibrium models. We believe that this result is helpful for making an economic assessment of the incentives in generation capacity investment in real world systems for the following reasons: in Section 2.2.2, we explain how we can solve a two-stage stochastic program as a nonlinear or linear optimization problem. Because of the availability of very efficient nonlinear programming solvers and decomposition methods for two-stage stochastic programs, solving a stochastic program is computationally much faster than solving a stochastic equilibrium model for large scale systems, which we also observe in our numerical experiments. However, this does not necessarily guarantee that there is an equivalent nonlinear or linear optimization problem for every equilibrium model. In only one case of operating-reserve pricing, we obtain an equilibrium problem that is not equivalent to an optimization problem.

- Under perfect competition, we show that single and two-stage representation of these "equilibrium" models are equivalent. In other words, the open loop equilibria and the closed loop equilibria of these models coincide. This result may not hold for electricity markets with strategic firms (see Wogrin et al. (2012)).

- Two-stage models with stochastic elements may prove computationally challenging to solve due to the resulting large scale problems (Kall and Wallace (2006)). Thus, we provide detailed algorithmic approaches for numerically tackling all these models. We utilize sample-path methods as well as the equivalence result in relation to solutions of stochastic equilibrium and optimization problems. Our computational methods allow solving large-scale problems with several uncertain parameters by utiliz-

[1] "equivalent" models or problems used in this thesis refer to problems or models having same solution sets.

ing deterministic off-the-shelf solvers. We also illustrate that these algorithmic approaches help further reduce the computational time.

- Through numerical experiments, we gain insights on the impact of demand uncertainty and to what extend these different mechanisms can remedy the resource adequacy issue. In particular, we first find that uncertainty of demand leads to higher total generation capacity expansion and a broader mix of technologies compared to the investment decisions assuming average demand levels. Furthermore for the same VOLL (or price cap) level, energy-only markets with VOLL pricing tend to lead to total generation capacity below the peak load with a certain probability whereas energy markets with a forward capacity market or operating-reserve pricing result in higher investments, assuming random demand with finite support and no forced outages. Finally, the regulator decisions (e.g., reserve capacity target) in capacity markets and operating-reserve pricing can be chosen in such a way that results in very similar investment levels and fuel mix of generation capacities in both market designs.

The result of the prevalence of stochastic programming for providing solutions to stochastic equilibrium models can be extended to generation capacity investment strategies in perfectly competitive electricity markets with different regulatory mechanisms such as emission trading scheme (Gürkan et al. (2012)) or renewable obligations (Gürkan and Langestraat (2012)). In addition, it may also be of interest to include risk averseness of generators by using "coherent risk measures" (e.g., Ehrenmann and Smeers (2011) and Ralph and Smeers (2011)). The resulting problems are still stochastic equilibrium problems which are somewhat modified versions of the stochastic equilibrium problems presented in this thesis. When the markets are "perfectly competitive" and "complete", the formulation of an equivalent two-stage stochastic program is still possible (Ralph and Smeers (2011)). This allows assessment of investment incentives of risk averse generators in "perfectly competitive" and "complete" markets by utilizing a two-stage stochastic program.

1.2.1.2 Strategic Generation Capacity Investments under Demand Uncertainty in Electricity Markets

Besides the impact of the market mechanism and imperfections mentioned in Section 1.2.1.1, investment incentives for generation capacity may be manipulated by electricity generators' capability to exercise market power. In energy-only competitive electricity spot markets where the supply bidding is at the sole short-run marginal cost, a generating firm (in particular a peak power plant) anticipating the impact of its capacity choice on market prices would likely exert market power and have incentive to underinvest in order to gain positive margins. In Chapter 4, we again model and analyze a two-stage problem of generation capacity investments but this time by strategic generators facing uncertain demand. These strategic generators compete as Cournot players when they choose their generation capacities under demand uncertainty at the first stage and they anticipate the impact of their capacity choice on perfectly competitive electricity market outcomes at the second stage. Thus, we have a closed loop model.

Due to strategic behavior, the two-stage capacity investment problem of each firm conveys different properties than the corresponding problem in Chapter 3 formulated under perfect competition. In the absence of market power, the corresponding two-stage capacity investment problem is a convex problem and, as shown in Chapter 3, most of the two-stage capacity investment equilibrium problems in competitive markets can be cast into convex non(linear) optimization problems similar to the early capacity expansion models. This result also suggests that an equilibrium of the closed loop model coincides with an equilibrium of the less complicated open loop model. However, convexity is in general lost when market power is introduced. The two-stage capacity investment problem of a strategic firm may be formulated as a bilevel problem, to be more precise as a stochastic MPEC (mathematical program with equilibrium constraints). When there are multiple strategic firms having an MPEC problem each, then one can speak of finding a solution to *an equilibrium problem with equilibrium constraints (EPEC)* (e.g., Gabriel et al. (2012)). It is well known that EPECs are in general nonconvex problems and they may have multiple or no solutions (e.g., Hu (2003), Ehrenmann (2004), Hu and Ralph (2007), Gabriel et al. (2012)). As a consequence, an equilibrium for the corresponding two-stage game may not exist or, if exists, it is in general not unique. Furthermore, when firms anticipate a perfectly competitive spot market outcome, an equilibrium of the closed loop model does not coincide with an equilibrium of a less complicated open loop model (see Wogrin et al. (2012)). Therefore, establishing a set of general conditions for existence and uniqueness is desirable for policy and market models including strategic firms so that users can have confidence in their solutions before solving such complicated closed loop models.

In Chapter 4, we establish sufficient conditions which guarantee the existence and uniqueness of a closed loop equilibrium in energy-only electricity markets with strategic generators. Our analysis include various types of such a closed loop model regarding the underlying price-demand relations, the assumed demand uncertainty with a broad class of distributions, and generators with symmetric or asymmetric costs. As mentioned before, we consider strategic firms facing uncertain demand and competing as Cournot players when they choose their generation capacities at the first stage while anticipating a perfectly competitive future spot market outcome based on their choices (for example they may expect regulatory intervention at the spot market). After a firm installs its capacity at the first stage, its production takes place in electricity spot market at the second stage which represents a competitive energy-only market. We look into both symmetric and asymmetric firms facing inelastic or elastic demand. In reality, electricity demand is uncertain and/or fluctuating with a very low elasticity. We focus on two cases of demand uncertainty: (i) an inelastic random demand drawn from a continuous distribution; or (ii) price-demand relationship that can be well approximated by a linear curve with a random intercept having a general underlying continuous probability distribution.

Short term aspects of market power (related to choices of production when capacities are given) within various specific market design assumptions have been extensively analyzed in the literature. Among these, equilibrium modeling of Nash games in quantities is a common approach (e.g., see Borenstein and Bushnell (1999), Wei and Smeers (1999), Hobbs (2001) for reviews). Long term aspects of market power related to generation capacity expansion have also received growing attention. Among the most important works in the literature, Kreps and Scheinkman (1983) analyze capacity choice at the first stage prior to price competition at the second stage (e.g., Bertrand) with deterministic demand. They show that there exists a unique equilibrium which is equivalent to the single stage Cournot outcomes. Gabszewicz and Poddar (1997) analyze capacity choices of two symmetric firms at the first stage prior to a Cournot market at the second stage with demand uncertainty. They prove existence of symmetric equilibrium and compare it with the deterministic solution of using expected demand. Murphy and Smeers (2005) move one step further and formulate open and closed loop market models with two asymmetric Cournot players and finite number of demand scenarios, each having a linear demand curve with respective probabilities. In their analysis, Murphy and Smeers (2005) conclude that the complexity of solving capacity expansion increases with a closed loop model even without considering the network limitations. They find closed loop examples which may not have an equilibrium. They show that if an equilibrium exists, then it is unique and falls between the perfect competition and open loop equilibria.

All the models mentioned above assume an elastic demand represented by a linear price-demand curve with random intercept and the structure of the assumed demand distribution is in general simple (i.e., uniform demand segments). Furthermore, their analysis are in general based on limited number of players (e.g., two players). In Chapter 4, we consider both inelastic random demand and an elastic demand represented by a linear price-demand curve with random intercept having a general underlying continuous probability distribution. In addition, we analyze the capacity choices of both symmetric and asymmetric firms. We establish existence and uniqueness results under a broader scope in terms of the underlying price-demand relations (elastic and inelastic demand), the assumed demand uncertainty with a broad class of continuous distributions, and any finite number of players with symmetric or asymmetric costs. We note that the continuity of demand distribution is a crucial assumption for establishment of our results.

In a similar closed loop model with symmetric firms and elastic demand, Grimm and Zoettl (2008) analyze strategic capacity choices under more general assumptions of a monotone random demand curve and its underlying probability distribution. They establish existence and uniqueness results when firms engage in Cournot competition at the second stage. They also show existence of equilibrium when firms anticipate competitive spot market outcomes at the second stage; however the uniqueness of equilibrium cannot be established under their general assumptions. In our analysis, the closed loop game with symmetric firms facing a linear demand curve with random intercept is indeed a subclass of their model whereas the closed loop model with inelastic random demand constitutes another type. For both cases, we establish sufficient conditions for the random demand's probability distribution which guarantee a unique equilibrium for symmetric firms anticipating competitive spot market outcomes. Furthermore, we give sufficient conditions for a unique equilibrium of such a closed loop game with asymmetric firms facing an elastic random demand (i.e., a linear demand curve with random intercept). We explain our analysis in more detail below and elaborate on our contributions.

- We show that whether firms sell their power via central auction or bilateral contracts in the spot market, the corresponding two-stage game yields the same equilibrium, if it exists. A similar result has also been established by Metzler et al. (2003) for a short-run Cournot competition in electricity spot markets where generation capacity is fixed. Metzler et al. (2003) show that a bilateral market model with Cournot producers and perfect arbitrageurs yields the same Nash-Cournot equilibrium as a Cournot competition in pool market. We extend their result for the two-stage capacity investment equilibrium problem we consider here. Our result indicates that when the spot market is perfectly competitive, different structures of buying and selling electricity do not affect the generation capacity choices of strategic firms. Thus, we establish the remaining of our results for the two-stage problem of each firm under the assumption of a stylized pool market model. In order to preserve analytical tractability, we do not consider any network constraints for the contributions below.

- In a single-stage Cournot model where symmetric firms choose their production output under uncertainty concerning the intercept of a linear price-demand function, Lagerlöf (2006) shows that if the distribution of the demand has a monotone hazard rate then the uniqueness of equilibrium is guaranteed. Chapter 4 establishes an extension of that result for a two-stage game where symmetric firms strategically choose their capacities at the first stage and their production takes place at the second stage in energy-only perfectly competitive electricity markets. Due to demand uncertainty, generators' capacity choices at first stage may not be utilized for all the demand realizations at the second stage. For the two-stage problem of each strategic firm maximizing its expected profit, the constraint set is closed and convex; however, the objective function is in general not concave. Hence, generalized concavity (such as strict logconcavity and strict quasiconcavity; see Chapter 2 for corresponding definitions) of the objective function are desirable properties for establishing existence and uniqueness results for the corresponding two-stage stochastic game. For such a two-stage game, we characterize the class of probability distributions under which we preserve generalized concavity (e.g., strict quasiconcavity) of each firm's first stage problem, thus we can establish the existence and uniqueness of equilibrium. We show that one general class of distributions are logconcave distributions (see Bagnoli and Bergstrom (2005)) having a monotone increasing hazard rate function under which the existence and uniqueness of equilibrium is guaranteed for symmetric generators facing an inelastic or elastic random demand. Thus, in case of symmetric firms, we can conclude that a large class of continuous probability distributions guarantee the uniqueness of equilibrium for the two-stage game.

- When firms are asymmetric, the first-stage expected profit functions of some firms (e.g., base-load generators) do not, in general, satisfy generalized concavity when demand is inelastic. We show by an example that even with two generators (e.g., peak-load and base-load) facing uniform demand, the first stage objective function of the base-load generator may not satisfy generalized concavity. Hence, an equilibrium may not exist. When demand is elastic (represented by a linear demand curve with random intercept), a unique equilibrium is guaranteed for another condition (different than logconcavity) for the continuous probability distribution of the intercept under which we prove strict concavity of all firms' expected profit functions. The corresponding condition is also similar to a standard assumption in the literature used for the demand function itself (e.g., Sherali et al. (1983), Wolf and Smeers (1997), Grimm and Zoettl (2008), and Xu (2005)).

- To sum up our contributions above, we consider various types of the two-stage investment model of firms regarding the underlying price-demand relations, the assumed demand uncertainty, and any finite number of players with symmetric or asymmetric costs. In most of the cases, we establish sufficient conditions under which solutions exist and are unique and we identify a broad class of commonly used continuous probability distributions of the random demand satisfying these conditions, which will enhance the value of such models for policy and market intelligence purposes. In particular, it is of interest to users to know the general sets of conditions which guarantee the existence and uniqueness of an equilibrium before solving such complicated closed loop models.

1.2.2 Optimal Threshold Levels in Stochastic Fluid Models via Simulation-Based Optimization

The research in manufacturing flow control can roughly be classified in two directions; papers directly addressing optimal control issues and papers that analyze and optimize the performance of plausible policies. Our work falls in the latter category but the policies we employ are strongly motivated by the research on the former.

Kimemia and Gershwin (1983) propose formal optimal control theory as an approach for studying general manufacturing flow control problems. They also propose a plausible class of control policies. The first formal proofs of optimality of a certain policy, however, were established later by Akella and Kumar (1986) and Bielecki and Kumar (1988) for a two machine state system with exponential up and down times, constant demand, linear holding and backorder costs. In particular, these papers establish the optimality of a hedging point policy. Whenever up, the machine should produce at full capacity below a target inventory level called the hedging point and produce at a rate which enables it to stay there once it reaches this target.

There are several extensions of the hedging point type optimality structure. For instance, for a multiple machine state system with a Markovian transition structure, a multiple hedging point type policy is optimal. For each machine state where production capacity exceeds demand rate, there is a corresponding hedging point (see Sethi et al. (2002) for related results). It is also known that with non-exponential machine state transitions, the structure of the optimal policy becomes more complex (see Hu and Xiang (1995) for an example) but hedging point policies remain useful due to their simplicity. In addition, if the production policy is required to depend only on the inventory level (and not on elapsed times since previous transitions), hedging point policies seem to be the only practical alternative for implementation. In turn, these type of policies are used not only for designing manufacturing controllers, but also for using capacity options and subcontracting strategies (e.g., Hu et al. (2004), Tan and Gershwin (2004)), and for coordinating production and marketing decisions (e.g., Zhang et al. (2001)).

There is also considerable literature on the performance analysis side for the implementation of hedging point policies. Sharifnia (1988) investigates the multiple machine state problem with Markovian transitions when a multiple hedging point policy is used and presents analytical results for the performance measures of the system. Finding the optimal hedging points, however, remains difficult. Liberopoulos and Hu (1995) also investigate the multiple machine state problem with Markovian transitions with a different focus. They establish useful monotonicity properties of the optimal hedging levels with respect to the machine state transition structure.

The particular manufacturing flow control problem that we consider in Chapter 5 is that of a single resource (a machine for simplicity) coping with a flow of demand whose rate varies randomly. The production capacity of the machine can take multiple values according to some external stochastic process. Similarly the demand rate at any given time is governed by another external and uncontrollable stochastic process. A hedging point policy, in general, requires determining a parameter corresponding to each machine and demand state. The corresponding parameter (hedging point) acts as an inventory target threshold in each machine/demand state. The machine should produce at the maximum rate if the inventory level is below the current threshold, should stop if the inventory level is above the current threshold, and should produce at the rate that would keep the inventory level at the current threshold level when it reaches there. The objective is then to find the optimal threshold (target inventory) levels minimizing expected inventory and backorder costs such that the production rate is adjusted in each state depending on the target level.

According to the previous studies in the literature, derivation of the closed-form analytical solution and finding the optimal hedging points are difficult for a multiple machine state problem even with Markovian transitions (e.g., Liberopoulos and Hu (1995), Sharifnia (1988)). If one is concerned with a system with several machine and demand states and fairly general random disturbances (i.e., not necessarily exponentially distributed), then the generality of the problem and the intractability of an analytical solution makes a simulation-based method an attractive approach. Using a simulation-based method for finding optimal hedging points has been tried before by Caramanis and Liberopoulos (1992), Liberopoulos and Caramanis (1994), Haurie et al. (1994), Yan et al. (1999), Brémaud et al. (1997), Yan et al. (1999), and Yin et al. (2001). However, the optimization side of all these papers is confined to the method of stochastic approximation and its variants, see Robbins and Monro (1951). Although very prominent, in practice these methods suffer from serious drawbacks such as slow convergence, lack of a good stopping criterion, and difficulty in enforcing feasibility. In particular, when there are constraints, these methods handle inequality constraints –even deterministic linear inequalities– via projection onto the feasible set. Such a projection operation can retard the performance of an optimization algorithm immensely, as illustrated by the simple example in Appendix 6 of Plambeck et al. (1996). Furthermore, even in the unconstrained case, the empirical performance of stochastic approximation type methods is very sensitive to the choice of a predetermined step size. Fu and Healy (1992), L'Ecuyer (1991), Glasserman and Tayur (1995), and Gürkan (2000) contain a number of examples which demonstrate this sensitivity.

These difficulties are partly reflected in the available numerical results of simulation-based optimization literature for finding optimal hedging points. None of the papers we are aware of (for single-part single-stage models) has numerical experiments with systems having more than two decision variables and they are all solving unconstrained problems. Furthermore, very often, comprehension of how the gradient estimates are derived is quite challenging for a non-specialist in gradient estimation literature since the arguments leading to a particular estimator are often done case by case. For example, Brémaud et al. (1997) is one of the papers developing a rigorous infinitesimal perturbation analysis (IPA) algorithm for a system whose transition times are not necessarily Markovian and the IPA analysis of the two hedging point case is done in a separate section than the IPA estimate for a single hedging point. Zhao and Melamed (2006, 2007) also deal with a general Make-to-Stock system with random demand and production rates and IPA derivatives are computed for the time averaged inventory level and time-averaged backorder level with respect to the base-stock level recursively via a sample-path analysis. However, for some initial conditions, derivation of their IPA derivatives is not straightforward, which is also mentioned by Melamed et al. (2010) as a drawback and there are no numerical experiments reported in these papers.

Application of IPA derivatives in stochastic fluid models is an active area of research. It is of interest to develop computationally robust IPA formulas which are flexible (i.e., free of number of control parameters and initial conditions of the system) and directly computable from a sample path (see Wardi et al. (2010), Melamed et al. (2010) for recent work), which we attain with our approach as mentioned below. Moreover, we also provide numerical experiments for various number of system control parameters and with service-level type probabilistic constraints, which indicates the flexibility and practicality of our methodology.

To overcome the difficulties encountered in the literature, we propose using a different simulation-based method, namely sample-path optimization, coupled with a generalized semi-Markov process (GSMP) representation of the system. As mentioned earlier, a GSMP can simply be thought as a mathematical framework that models the evolution of a discrete-event simulation representation of the system (see Glynn (1989) and Shedler (1993)). As we summarize, this has the following consequences which are the main contributions of Chapter 5:

- On the modeling and gradient estimation side, the GSMP approach allows us to derive a rigorous, intuitive, flexible, and transparent algorithm for a general system that can compute the function and gradient values of the objective function and service level-type probabilistic constraints at any parameter setting in a single simulation run.

- Using sample-path optimization we avoid certain drawbacks associated with stochastic approximation type methods such as having to fine tune several parameters, slow convergence, and ad-hoc handling of constraints. Thus on the optimization side, we are able to solve much larger (with several machine and demand states) and general (not necessarily exponential disturbances) problems even with probabilistic constraints.

- We report numerical results for systems with more than twenty hedging points and service-level type probabilistic constraints. In these numerical studies, our method performs quite well even on problems with nonconvex objective functions and probabilistic constraints, which are considered very difficult to solve by current standards. In addition, we obtain insights on the structure of optimal hedging points for much larger systems than is usually reported in the literature. In particular, we observe that production and demand variability has an important effect on the optimal threshold levels of a system. As the production and demand variability increases, the optimal threshold levels also increase to hold enough safety stock against the uncertainty in production and demand rates. Furthermore, as the utilization of a system converges to one, the optimal threshold levels get close to each other and converge to a single threshold level.

Finally, our general approach with appropriate modifications could be applicable to several other challenging problems encountered in practice. These problems can also be formulated as stochastic fluid models that involve the optimization of a finite number of threshold levels as a control parameter of certain buffer quantities in the system. Examples in the literature include production/marketing problems involving dynamic pricing where the optimal price may vary as a function of threshold (e.g., Zhang et al. (2001)); subcontracting/manufacturing problems where subcontracting strategies depend on the inventory level, for example using a subcontractor when the number of waiting orders exceeds a certain threshold (e.g., Tan (2002a), Hu et al. (2004), and Tan and Gershwin (2004)). Several admission control or routing problems in telecommunication networks and call centers are also among the implementations of stochastic fluid models, e.g., admit new arrivals of a given class if the buffer is lower than a threshold, reject otherwise (e.g., Sun et al. (2004) and Liu and Gong (2002)). For the interested reader, Cassandras (2007) provides an overview of the theory of stochastic fluid models and their applications with threshold-based controllers in various fields such as finance, telecommunications, and engineering.

1.3 Outline of the Thesis

The thesis is organized as follows: Chapter 2 gives the necessary background information for the mathematical formulations and the notation used throughout this thesis. The common formulations of optimization and equilibrium problems and the relation between them are summarized briefly in this chapter as well as an overview of the sample-path methods for solving stochastic optimization and equilibrium problems is given with references to the literature. The remaining chapters of this thesis are based on the following papers:

Chapter 3: Generation Capacity Investments in Electricity Markets: Perfect Competition
G. Gürkan, Ö. Özdemir, Y. Smeers
Working Paper, Tilburg University, Tilburg, The Netherlands, 2012 (Gürkan et al. (2012a)).

Chapter 4: Strategic Generation Capacity Choice under Demand Uncertainty: Analysis of Nash Equilibria in Electricity Markets
G. Gürkan, Ö. Özdemir, Y. Smeers
Working Paper, Tilburg University, Tilburg, The Netherlands, 2012 (Gürkan et al. (2012b)) .

Chapter 5: Optimal Threshold Levels in Stochastic Fluid Models via Simulation-based Optimization
G. Gürkan, F. Karaesmen, Ö. Özdemir
Discrete Event Dynamic Systems (2007) 17, 53-97 (Gürkan et al. (2007)).

Note that Chapter 5 is a slightly adjusted version of Gürkan et al. (2007) in which the abstract and the introduction are updated with some recent literature in this area and with the most important insights observed from the numerical experiments performed in this study.

Chapter 2

Preliminaries

This chapter gives an introduction to mathematical formulations and notation used throughout this thesis. It mainly focuses on optimization and equilibrium models to model the problems of one or several decision makers. A problem itself may be stochastic if the decisions are made under uncertainty. Section 2.1 starts with basic definitions of the common terminology used in this thesis. In Section 2.2, we introduce standard deterministic and stochastic optimization models in operations research that are used to formulate a single agent's problem. Then, in Section 2.3, we give a brief overview of the concepts of a game and an equilibrium which are commonly encountered in problems where there are multiple agents interacting with each other in a noncooperative fashion. We also introduce some typical mathematical models used to formulate the equilibrium problem of multiple agents. Finally, Section 2.4 gives an overview of simulation-based methods, so called *sample-path methods* (also known as *sample average approximation methods*), which are used as solution methods in numerical applications of the stochastic problems presented throughout the thesis.

2.1 Definitions

2.1.1 Generalizations of Concave/Convex Functions

Generalized concave/convex functions are important for the characterization of a solution in optimization problems because they are among the few functions for which sufficient optimality criteria can be given. Similarly, they are also crucial for equilibrium problems since an equilibrium is a solution that is reached among all agents maximizing their utilities subject to a set of constraints. We first start with the definitions of concave and convex functions. Then we give brief introductions to generalized concave and convex functions (see Bazara et al. (2006) for details), which are mainly referred in Chapter 4 of this thesis.

Definition 2.1.1 (Concavity/Convexity)**.** Let X be a nonempty convex set in $\mathscr{R}^n$.

- The function $f : X \to \mathscr{R}$ is concave on X if $f(\lambda x_1 + (1-\lambda)x_2)) \geq \lambda f(x_1) + (1-\lambda) f(x_2)$ for each

$x_1, x_2 \in X$ and $\lambda \in (0,1)$. The function f is strictly concave on S if the inequality holds as a strict inequality for each $x_1, x_2 \in X$ with $x_1 \neq x_2$ and for each $\lambda \in (0,1)$,

- The function $f : X \to \mathscr{R}$ is convex on X if $f(\lambda x_1 + (1-\lambda)x_2)) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$ for each $x_1, x_2 \in X$ and $\lambda \in (0,1)$. The function f is strictly convex on X if the inequality is true as a strict inequality for each $x_1, x_2 \in X$ with $x_1 \neq x_2$ and for each $\lambda \in (0,1)$.

Notice that a continuous concave function can have flat sections, linear portions, and kinks. A strictly concave continuous function can have kinks, but it can have neither flat sections nor linear portions. A strictly concave function is also a concave function while the converse is not true. The same is true for convex functions. The following definition introduces the most inclusive class of generalized concave/convex functions.

Definition 2.1.2 (Quasiconcavity and Quasiconvexity)**.** Let X be a nonempty convex set in $\mathscr{R}^n$.

- The function $f : X \to \mathscr{R}$ is quasiconcave on X if the following inequality is true for each $x_1, x_2 \in X$ and $\lambda \in (0,1)$:

  $$f(\lambda x_1 + (1-\lambda)x_2)) \geq \min\{f(x_1), f(x_2)\}.$$

  The function f is strictly[1] quasiconcave on X if the inequality holds as a strict inequality for each $x_1, x_2 \in X$ with $x_1 \neq x_2$ and for each $\lambda \in (0,1)$.

- The function $f : X \to \mathscr{R}$ is quasiconvex on X if the following inequality is true for each $x_1, x_2 \in X$ and $\lambda \in (0,1)$:

  $$f(\lambda x_1 + (1-\lambda)x_2)) \leq \max\{f(x_1), f(x_2)\}.$$

  The function f is strictly[1] quasiconvex on X if the inequality holds as a strict inequality for each $x_1, x_2 \in X$ with $x_1 \neq x_2$ and for each $\lambda \in (0,1)$.

A function f is called (strictly) quasiconvex if and only if $-f$ is (strictly) quasiconcave. Let's focus on the quasiconcave functions in Definition 2.1.2. A function f is quasiconcave if whenever $f(x_2) \leq f(x_1)$, f is grater than equal to $f(x_1)$ at all convex combinations of x_1 and x_2. Hence, if f decreases its value at a point along any direction, it must remain nonincreasing in that direction. A continuous quasiconcave function can have flat, linear, and even convex portions; however, when it starts to decrease it cannot increase again (i.e., the function cannot have multiple peaks). Strict quasiconcavity rules out the flat sections, but can still contain linear and convex portions. Note that a strictly concave or a strictly quasiconcave function is also quasiconcave while the

[1]We caution the reader that the same definition is sometimes referred to in the literature as "strongly" quasiconcave/quasiconvex.

converse is not true. Similar results hold for quasiconvex functions.

Strictly quasiconcave and strictly quasiconvex functions are especially important in nonlinear programming because they ensure that a local maximum and a local minimum over a convex set are a unique global maximum and a unique global minimum, respectively (see Theorem 3.5.9 of Bazara et al. (2006)). However, strictly quasiconcave (strictly quasiconvex) functions do not share the particular property of concave (convex) functions which says a zero gradient implies the function is at a global maximum (global minimum) (see Definition 3.5.9. of Bazara et al. (2006)).

There is another class of functions, so called pseudoconcave (pseudoconvex) functions, which is very close to the class of quasiconcave (quasiconvex) functions and it retains the property of concave (convex) functions that its gradient is never zero except at its global maxima (minima) (see Definition 3.5.10. of Bazara et al. (2006)).

Definition 2.1.3 (Pseudoconcavity and Pseudoconvexity)**.** Let $f : X \to \mathscr{R}$ be a differentiable function defined over the open nonempty convex set $X \subset \mathscr{R}^n$.

- f is pseudoconcave if for each $x_1, x_2 \in X$ with $\nabla f(x_1)^T(x_2 - x_1) \leq 0$, it holds that $f(x_2) \leq f(x_1)$; or equivalently, if $f(x_2) > f(x_1) \Rightarrow f(x_1)^T(x_2 - x_1) > 0$ for each $x_1, x_2 \in X$.

- f is pseudoconvex if for each $x_1, x_2 \in X$ with $\nabla f(x_1)^T(x_2 - x_1) \geq 0$, it holds that $f(x_2) \geq f(x_1)$; or equivalently, if $f(x_2) < f(x_1) \Rightarrow f(x_1)^T(x_2 - x_1) < 0$ for each $x_1, x_2 \in X$.

The function f is (strictly) pseudoconvex if and only if $-f$ is (strictly) pseudoconcave. (Strictly) pseudoconcave functions are also (strictly) quasiconcave while the converse is not true. Another generalized function referred in the thesis is logconcave functions, which is defined next.

Definition 2.1.4 (Logconcavity)**.** Let $f : X \to \mathscr{R}$ be a nonnegative function defined over the open convex set $X \subset \mathscr{R}^n$. f is logconcave if it satisfies the inequality

$$f(\lambda x_1 + (1-\lambda)x_2)) \geq f(x_1)^{\lambda} f(x_2)^{1-\lambda} \text{ for each } x_1, x_2 \in X \text{ and } \lambda \in (0,1).$$

Note that a positive function f is logconcave if its logarithm ($\log f$) is concave. Every (strictly) concave function that is nonnegative on its domain is (strictly) logconcave. Moreover, (strict) log-concavity implies (strict) quasiconcavity as shown by Avriel (1972).

We end this section by summarizing the relationships between the various forms of concavity in Figure 2.1 (see Boyd and Vandenberghe (2004) and Bazara et al. (2006) for the details). According to Figure 2.1, the following relations hold which will be relevant for Chapter 4:

- f is (strictly) concave $\Rightarrow$ f is (strictly) quasiconcave.

- f is differentiable and (strictly) concave $\Rightarrow$ f is (strictly) pseudoconcave $\Rightarrow$ f is (strictly) quasiconcave.

- f is nonnegative and (strictly) concave $\Rightarrow$ f is (strictly) logconcave $\Rightarrow$ f is (strictly) quasiconcave.

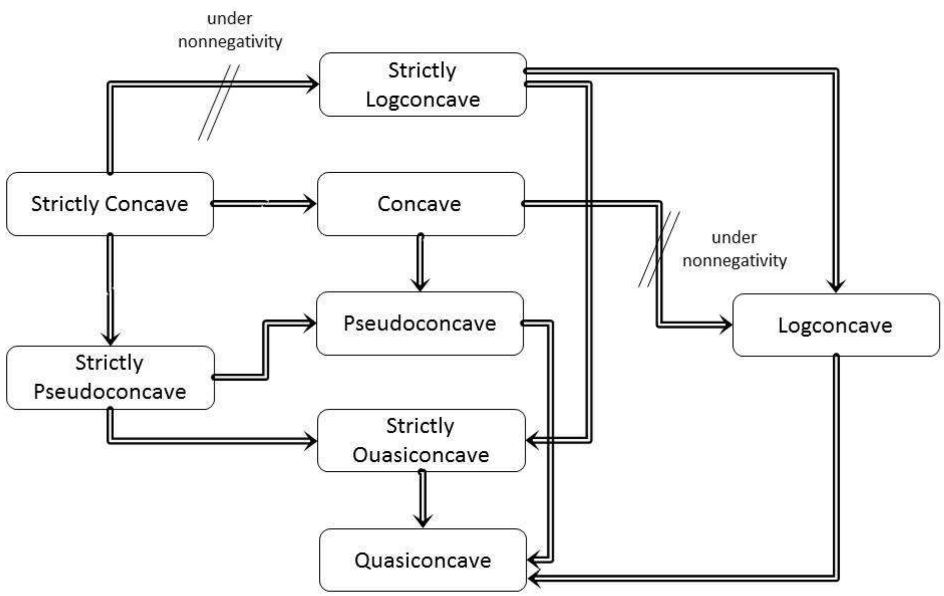


Figure 2.1: Relationships between the various forms of generalized concavity (e.g., Boyd and Vandenberghe (2004), Bazara et al. (2006))

2.1.2 Monotonicity

Definition 2.1.5. The continuous mapping $F : \mathscr{R}^q \mapsto \mathscr{R}^q$ is
(a) monotone over a set B if

$$[F(x_1) - F(x_2)]^T (x_1 - x_2) \geq 0,\ \forall x_1, x_2 \in B;$$

(b) strictly monotone over B if

$$[F(x_1) - F(x_2)]^T (x_1 - x_2) > 0,\ \forall x_1, x_2 \in B,\ x_1 \neq x_2;$$

(c) strongly monotone over B if there exists an $\alpha > 0$ such that

$$[F(x_1) - F(x_2)]^T (x_1 - x_2) \geq \alpha \, ||x_1 - x_2||^2,\ \forall x_1, x_2 \in B.$$

Note that when F is continuously differentiable and if the Jacobian matrix ∇F of F is positive semidefinite (positive definite) then F is monotone (strictly monotone). Moreover, the following relations are valid:

$$\text{strong monotonicity} \Rightarrow \text{strict monotonicity} \Rightarrow \text{monotonicity}.$$

2.2 A Single Agent's Problem

In this section, we introduce two types of modeling tools to formulate the decision making process of a single agent. In Section 2.2.1, we start with a special class of optimization problems called *nonlinear programs* (NLPs) and their stationary conditions, namely *Karush-Kuhn-Tucker* (KKT) conditions. A standard NLP formulation is often used to formulate single-stage optimization problems. Whereas the deterministic optimization problems are formulated with fixed or known parameters, most real world optimization problems include parameters which vary or are unknown at the time a decision should be made. In Section 2.2.2, we introduce *two-stage stochastic programs* which can be used for modeling a single agent's optimization problem that involve uncertainty in some parameters.

2.2.1 Nonlinear Programming (NLP)

Mathematical programming deals with the optimization of an objective function in the presence of inequality and equality constraints. If the objective function and all the constraints are linear then it is a *linear program (LP)*. Otherwise, it is a *nonlinear program (NLP)*. The development of state-of-art solvers and high-speed computers has made LPs and most of the NLPs widely used tools for solving problems in diverse fields. Although traditional (non)linear programming deals with deterministic optimization, the mathematical theory of (N)LPs can also serve as a basis for the analysis and solution of stochastic optimization problems. For instance, the state-of-art solvers for (N)LPs may be used to solve complex stochastic problems, which we explain in more detail in Section 2.4. For solving some of the problems presented in Chapters 3 and 5 in this thesis, we use stochastic optimization and resort to the existing theory and efficient solution methods of (N)LPs. Next, we give a brief overview of some known results from the theory of (N)LPs which are also utilized in this thesis.

We consider the following constrained (non)linear program

$$\begin{aligned} \min \quad & f(x) & \\ \text{s.t.} \quad & g(x) \geq 0 & (\lambda) \\ & h(x) = 0 & (\mu) \\ & x \geq 0 & (\nu), \end{aligned} \tag{2.1}$$

where $x \in \mathscr{R}^n$ is the vector of decision variables and $f : \mathscr{R}^n \mapsto \mathscr{R}$, $g : \mathscr{R}^n \mapsto \mathscr{R}^k$ and $h : \mathscr{R}^n \mapsto \mathscr{R}^m$ are continuously differentiable functions. We will focus on finding a stationary point that satisfies the first order optimality conditions of the NLP (2.1). The Lagrangian theory plays a crucial role in defining the first order optimality conditions of a (non)linear program. For the NLP (2.1), the Lagrangian function is defined as

$$L(x,\lambda,\mu,\nu) := f(x) - \lambda^T g(x) - \mu^T h(x) - \nu^T x,$$

where $\lambda \in \mathscr{R}^k_+$, $\mu \in \mathscr{R}^m$, and $\nu \in \mathscr{R}^n_+$ are the vector of Lagrange multipliers (dual variables) corresponding to the inequality, equality, and nonnegativity constraints, respectively.

If x^* is a local minimizer of NLP (2.1) and a regularity condition such as the *linear independence constraint qualification (LICQ)* or *Slater constraint qualification* (see Bazara et al. (2006) for these and other types of constraint qualifications) holds at x^*, then there exists λ^*, μ^*, and ν^* such that

$$\begin{aligned} \nabla_x L(x^*,\lambda^*,\mu^*,\nu^*) &= 0 \\ 0 \le x^* \perp \nu^* &\ge 0 \\ 0 \le g(x^*) \perp \lambda^* &\ge 0 \\ h(x^*) &= 0, \end{aligned} \tag{2.2}$$

where we use the notation "$\perp$" to signify that the inner product of the two variables equal to zero (e.g., $x^{*T}\nu^* = 0$). Note that $L\nabla_x L(x^*,\lambda^*,\mu^*) = \nabla_x f(x^*) - \lambda^{*T}\nabla_x g(x^*) - \mu^{*T}\nabla_x h(x^*) - \nu^*$. Thus,
(i) if $x_i^* = 0$, then $\nabla_{x_i} f(x^*) - \lambda^{*T}\nabla_{x_i} g(x^*) - \mu^{*T}\nabla_{x_i} h(x^*) = \nu_i^* \ge 0$,
(ii) if $x_i^* > 0$, then $\nabla_{x_i} f(x^*) - \lambda^{*T}\nabla_{x_i} g(x^*) - \mu^{*T}\nabla_{x_i} h(x^*) = \nu_i^* = 0$.

As a result of *(i)* and *(ii)*, we can eliminate ν^* and rewrite the first order optimality conditions in (2.2) as follows:

$$\begin{array}{lcl} 0 \le \nabla_x f(x^*) - \lambda^{*T}\nabla_x g(x^*) - \mu^{*T}\nabla_x h(x^*) & \perp & x^* \ge 0 \\ 0 \le g(x^*) & \perp & \lambda^* \ge 0 \\ h(x^*) = 0. & & \end{array} \tag{2.3}$$

The conditions in (2.3) together with the regularity condition on the point x^* establish the necessary optimality conditions for the NLP (2.1) and are called Karush-Kuhn-Tucker (KKT) conditions. Under certain conditions, the KKT conditions are also sufficient to find an optimal solution of the NLP (2.1). Proposition 2.2.1 gives example for which the KKT conditions are sufficient optimality conditions; hence the solution of (2.3) and the solution of the NLP (2.1) are equivalent. The (non)linear programs we formulate in Chapter 3 fall also in this category.

Proposition 2.2.1. *Let f be pseudoconvex, g be a vector-valued quasiconcave function, and h be a vector-valued affine function. If there exits (x^*,λ^*,μ^*) such that the KKT conditions are satisfied, then x^* is a global minimum. A special case is when (2.1) is a linear program (LP) where f,g, and h are linear functions of x.*

Note that in Proposition 2.2.1, x^* satisfying the KKT conditions is a global minimum but not necessarily unique. When the functions f and g are twice continuously differentiable, if the Hessian matrix of the Lagrangian function is positive definite at (x^*, λ^*, μ^*), then the KKT conditions are also sufficient for x^* to be a unique minimizer of the NLP (2.1). For more extensive theory in nonlinear programming, the reader is referred to Simone and Blume (1994) and Bazara et al. (2006).

2.2.2 Two-stage Stochastic Programming (SP)

The basic idea of two-stage stochastic programming is that the optimal decisions are made before the realization of the uncertain data, $\xi(\omega)$, and do not depend on future observations. In a standard two-stage stochastic program, one is interested in finding an optimal decision x^* which minimizes an expected value function $E_\omega[F(\xi(\omega), x)]$ (e.g., expected cost) at the first stage. This expected value function represents the expectation of a random function $F(\xi(\omega), x)$ (defined on a common probability space $(\Omega, \mathscr{F}, P)$) which is an optimal value of a second-stage problem after realization of the uncertain data $\xi(\omega)$. To be more precise, the standard formulation of a two-stage stochastic program is given by

$$x^* = \arg\min_{x \in X} E_\omega[F(\xi(\omega), x)], \tag{2.4}$$

where $X \in \mathscr{R}^n$, $\xi(\omega)$ is a random vector, and the random function $F(\xi(\omega), x)$ is the optimal value of the second stage optimization problem at given x and $\omega \in \Omega$:

$$F(\xi(\omega), x) := \min_{y(\omega)} \{ f(\xi(\omega), x, y(\omega)) \text{ s.t. } g(\xi(\omega), x, y(\omega)) \geq 0, h(\omega, x, y(\omega)) = 0 \}.$$

Here, $x \in \mathscr{R}^n$ is the first stage decision vector which does not depend on $\xi(\omega)$ and $y(\omega) \in \mathscr{R}^m$ is the second stage decision vector which takes different values with different realizations of $\xi(\omega)$. Note that x is a parameter to the second stage problem. Although the value of y depends on the value of all parameters (including x) of the second stage problem, we emphasize its dependency with respect to $\xi(\omega)$ in the notation in order to facilitate the main distinction between first and second stage variables.

$\xi(\omega)$ may have a discrete probability distribution with finite number of possible outcomes (i.e., $\xi_1, \xi_2, \ldots, \xi_S$) and respective (positive) probabilities $\pi_s, s = 1, 2, \ldots, S$. However, $\xi(\omega)$ may also be a vector of random parameters with continuous probability distributions. Furthermore, the corresponding probability distributions of $\xi(\omega)$ may or may not be known. It may be known in situations where realizations repeat themselves several times and the distribution can be accurately estimated from historical data. However, without knowing the exact distribution of the random variable, samples of random parameters may be obtained simply from historical data.

If $\xi(\omega)$ has a discrete probability distribution with a finite number of scenarios then one may compute a solution to the stochastic programming problem by solving the equivalent deterministic optimization problem. When the probability distribution of $\xi(\omega)$ is continuous, it is most of the time impossible to calculate the expected value $E_\omega[F(\xi(\omega),x)]$ analytically. One of the solution methodologies replaces the random variables by a finite random sample generated beforehand and solves the equivalent (deterministic) optimization problem by a mathematical programming method (as one would do for the finite scenario case), which is often known as external sampling (Shapiro (2003)). The method we utilize in this thesis is called *"sample-path optimization"* or *"sample average approximation method"* which uses external sampling and state-of the art deterministic solvers to obtain an approximate of the solution to a two-stage stochastic problem. The method is described in detail in Section 2.4. In Chapter 3, we formulate a two-stage stochastic program of a central decision maker in decentralized electricity markets and we utilize a *sample-path optimization method* to solve the corresponding stochastic problem.

2.3 Noncooperative Decision Making in Multi-agent Systems

When there are multiple agents (e.g., players) who minimize (maximize) their own costs (profits) and whose strategies have impact on the other's decisions, we have a competition in a noncooperative manner. We call the collection of solution strategies which optimize all the players' problems simultaneously so that no one has incentive to deviate from his/her strategy an *equilibrium* and the corresponding problem an *equilibrium problem*. One of the classical concepts of equilibrium in noncooperative game theory is the *"Nash equilibrium"*, which we discuss briefly in Section 2.3.1. We introduce two types of competition in Section 2.3.2, namely perfect competition and imperfect competition. We give an example of a perfect competition in a market with players consisting of suppliers and consumers. This example can be viewed as a simpler formulation of each power supplier's problem that we analyze in Chapter 3. We then introduce the concept of imperfect competition with an example of Cournot competition. The main distinction between perfect and imperfect competition is that in imperfect competition at least one player has market power. The concept of market power is discussed in more detail in Chapter 4 where we analyze the strategic capacity choice of a power supplier in a decentralized electricity market under imperfect competition.

Mixed complementarity problems (MCPs) or *variational inequality (VI) problems*, which we introduce in Section 2.3.3, are extensively used to model and solve economic equilibrium problems which arise from competition of multiple players in a market (see Ferris and Pang (1997), Gabriel et al. (2012) for an extensive overview). In addition, *bilevel problems*, mentioned in Section 2.3.4, can be used to model two-stage games where there may be multiple agents at both stages. In Chapters 3 and 4, we also utilize MCPs and bilevel problems for modeling in the context of competition in decentralized electricity markets.

2.3.1 Nash Games

A game is a model of interacting players. Each player has a set of possible actions. The model captures interaction between the players by allowing each player to be affected by the actions of all players. Players can interact in a variety of ways (e.g., by choosing prices or quantities in sequential or simultaneous way) and many of these have been studied by using game theory (e.g., Varian (2006)). Each player chooses an action, which is the best for him/her, not having been informed of the other players' actions. As mentioned before, the collection of solution strategies (actions) which are optimal for all the players' problems is an *equilibrium*. Even though players are not informed of others' actions, they may anticipate what they will be, e.g., from their past experience. In this section, we give a brief definition of *"Nash equilibrium"* in which each player chooses his/her optimal strategy given the optimal strategies of other players, thus no one has incentive to deviate from his/her strategy.

The concept of equilibrium point in an N-person game, so called *"Nash equilibrium"*, was introduced by John Forbes Nash (Nash (1950)). In noncooperative game theory, the Nash equilibrium is a solution concept of a game involving two or more players, in which no player has an incentive to change his/her own strategy assuming that the other players do not change their decisions. In other words, if each player has chosen a strategy and no player can benefit by changing his/her strategy while the other players keep theirs unchanged, then the current set of strategy choices and the corresponding payoffs (costs or profits) constitute a Nash equilibrium.

Suppose that there are N players and player i's strategy set is X_i and is independent of the other players' (his/her rivals) strategies. Player i's cost function $f_i(x^i, \mathbf{x}^{-\mathbf{i}})$ depends on all players' strategies. x^i denotes the vector of player i's own strategy and $\mathbf{x}^{-\mathbf{i}} \equiv (x^j)_{j \neq i}$ denotes the vector of his/her rivals' strategies. For each fixed but arbitrary strategy $\mathbf{x}^{-\mathbf{i}}$, player i determines his/her optimal strategy x^{*i} by solving the cost minimization problem:

$$\min_{x^i} \{ f_i(x^i, \mathbf{x}^{-\mathbf{i}}) \text{ s.t. } x^i \in X_i \}. \tag{2.5}$$

For any $\mathbf{x}^{-\mathbf{i}}$, let $\Upsilon_i(\mathbf{x}^{-\mathbf{i}})$ denote the solution set of the problem (2.5). Then a Nash equilibrium x^* is a point such that $x^{*i} \in \Upsilon_i(x^{*-i})$ for all i. There is an extensive literature which provides various results on the existence of a Nash equilibrium. One of the well known results is that Nash games possess an equilibrium if (1) the strategy spaces are nonempty, convex, and compact, and (2) players have continuous and quasiconcave (quasiconvex) payoff (cost) functions (Fudenberg and Tirole (1991)). This result introduced by Debreu (1952) is given in the following proposition.

Proposition 2.3.1. (Debreu (1952)) *An N-person strategic game has a Nash equilibrium if the strategy sets $X_i, i = 1 \ldots N$, in this game are non-empty, compact, convex subsets of a Euclidian space and the payoff (cost) functions f_i are continuous in x and quasiconcave (quasiconvex) in x^i where x^i is the strategy of player i.*

Under certain conditions, the Nash equilibrium x^* can be computed by solving a variational inequality (VI) or a mixed complementarity problem (MCP). In Section 2.3.3, we introduce these problems and briefly explain how we can formulate Nash games and compute Nash equilibria via VI or MCP formulations. We also give a brief overview of existence and uniqueness theorems within the context of these formulations.

Finally, we define above a standard Nash game where each player has a feasible set of strategies that is independent of the other players' strategies. As an extension of the standard Nash game, if each player i's strategy set X_i depends on his/her rivals' strategies, then we speak of *generalized Nash games*. In generalized Nash games, we have $X_i(\mathbf{x^{-i}})$ as the feasible strategy set for player i and the equilibrium is called *generalized Nash equilibrium*. It is well known that generalized Nash equilibria are typically not unique. This result is based on the observation that the Lagrange multiplier (shadow price) for the joint active constraints[2] in $X_i(\mathbf{x^{-i}})$ can be different for each player. If the shadow prices for the joint constraints are taken to be common for each player, then the generalized Nash equilibrium problem can be reduced to a VI problem. Generalized Nash equilibrium problems are out of the scope of this thesis. Interested readers are referred to Harker (1991) for detailed analysis.

2.3.2 Perfect vs Imperfect Competition

In Chapters 3 and 4, we analyze the generation capacity investments in decentralized electricity markets under perfect and imperfect competition, respectively. In order to clarify the concept of perfect and imperfect competition used in this thesis, we next give a simple example for each case in a market where there are suppliers and consumers. Suppliers are assumed to compete in quantities. Each firm as a supplier decides on their production level and the market sets the price at which the quantity demanded by all consumers are satisfied by the quantity supplied by all producers, resulting in an economic equilibrium of price and quantity.

2.3.2.1 Perfect Competition

A market is perfectly competitive if each player assumes that the market price is independent of its own action (Varian (2006)). Thus in a perfectly competitive market, players are price-takers; that is they act as if the market price is given and their actions do not influence the price. Perfect competition might be a reasonable assumption if the industry is composed of many suppliers that produce an identical product and each firm has a small share in the market (Varian (2006)). For instance, in a perfectly competitive market with firms competing in production quantities of an identical product, each firm does not anticipate (or worry about) how it can influence the market price by changing its supply. It is a price-taker and all it worries about is how much output it wants to produce to maximize its profit. Whatever it produces can be sold at market price.

Let x^i be the vector of supply levels of a price-taking firm i, then (2.6) gives an example of the profit maximization problem of such a firm:

$$\begin{aligned} \max_{x^i} \quad & p^T x^i - C_i(x^i) \\ & \text{s.t. } x^i \in X_i, \end{aligned} \tag{2.6}$$

where X_i is the strategy set of firm i, $C_i(x^i)$ is the continuously differentiable production cost function of firm

[2]The common constraints, in each player's strategy set $X_i(\mathbf{x^{-i}})$, that are binding at equilibrium.

i, and p is the vector of unit prices of x^i which is a fixed (exogenous) parameter in firm i's problem. Although the market price p is exogenous to firm i's problem, it is endogenous to the whole system and it is set at a level where the market clears itself; that is demand equals the total quantity produced by all firms. Furthermore, if X_i is a convex set and $C_i(x^i)$ is a convex function, we have a convex optimization problem for each firm. Then by using Proposition 2.2.1 in Section 2.2.1, the KKT conditions of problem (2.6) for each firm i are the necessary and sufficient optimality conditions. As p is a fixed parameter in firm i's problem, while formulating its KKT conditions, firm i's marginal profit, which is the gradient of the objective function, is computed as follows:

$$MP_i(x^i) = p - \nabla_{x^i} C_i(x^i), \tag{2.7}$$

where the equilibrium values of p and x_i will be determined by solving the KKT conditions of all firms and market clearance conditions simultaneously. Below we compare this marginal profit with the marginal profit of a firm in imperfect competition. The use of the KKT conditions in solving the economic equilibrium in a market with several players will be explained in more detail in Section 2.3.3.

2.3.2.2 Imperfect Competition

In imperfect competition, at least one player exerts market power; that is he/she behaves strategically and anticipates the impact of his/her decisions on the market equilibria (i.e., total supply and market prices). There are several types of imperfect competition. In this section, we consider one case, namely *Cournot games*, which is relevant for the content and discussions in this thesis. In *Cournot games*, all players exert market power while competing in quantities simultaneously.

Cournot games: In a Cournot game, each firm as a supplier takes the quantity set by its competitors as given and then chooses its profit maximizing quantity x^{*i} by anticipating the impact of his decision x^i on the market price. (2.8) gives an example of the profit maximization problem of a strategic firm i choosing its optimal quantity x^{*i}:

$$\begin{aligned} \max_{x^i} \quad & P(x^i, \mathbf{x^{-i}})^T x^i - C_i(x^i) \\ & \text{s.t. } x^i \in X_i, \end{aligned} \tag{2.8}$$

where X_i is the strategy set of firm i, $C_i(\cdot)$ is the continuously differentiable production cost function of firm i, $\mathbf{x^{-i}}$ is the vector of quantities supplied by firm i's rivals, and $P(\cdot, \mathbf{x^{-i}})$ is the vector valued market price function at given $\mathbf{x^{-i}}$.

By using (2.3) in Section 2.2.1, the firm i's first order optimality conditions will contain the following gradient of its objective function for given $\mathbf{x^{-i}}$:

$$MP_i(x^i, \mathbf{x^{-i}}) = \nabla_{x^i} P(x^i, \mathbf{x^{-i}})^T x^i + P(x^i, \mathbf{x^{-i}}) - \nabla_{x^i} C_i(x^i), \tag{2.9}$$

where MP_i is the firm i's marginal profit. Note that this is different from perfect competition since firm i behaves strategically, its marginal profit contains the gradient of the price as well. If problem (2.8) is a convex optimization problem for each firm then one can find the corresponding Nash-Cournot equilibrium by solving the KKT conditions of all the players and the market clearance conditions simultaneously. In Section 2.3.3, we briefly explain how one can solve Nash games which also includes Cournot players by using the KKT conditions. Finally, it is a well known result that Cournot games in the limit (as number of firms grows to infinity) result in perfect competition equilibria (Varian (2006)).

2.3.3 Mixed Complementarity Problems (MCPs) and Variational Inequalities (VIs)

A variety of physical and economic phenomena including economic equilibrium problems are most naturally modeled by *mixed complementarity problems (MCPs)* in the sense that certain pairs of inequality constraints must be complementary and at least one must hold with equality. We next give the standard formulation of an MCP and present some basic results regarding the existence and uniqueness of its solution. Then, we briefly mention some of the application areas of MCPs relevant to this thesis such as finding a solution to an NLP or an equilibrium to a Nash game.

Given lower bounds l_i and upper bounds u_i with $-\infty \leq l_i < u_i \leq \infty$ for all $i \in \{1,2,\dots,q\}$, the standard MCP is defined as the problem of finding a point $z^* \in \mathscr{R}^q$ inside the box $B = \{z \mid l \leq z \leq u\}$ that is *complementary* to the nonlinear function $F : \mathscr{R}^q \mapsto \mathscr{R}^q$. The point z^* is *complementary* to $F(z^*)$ when

$$\begin{array}{rlll} \text{either} & z_i^* = l_i & \text{and} & F_i(z^*) \geq 0 \\ \text{or} & z_i^* = u_i & \text{and} & F_i(z^*) \leq 0 \quad \text{for} \quad i = 1,2,\dots,q, \\ \text{or} & l_i < z_i^* < u_i & \text{and} & F_i(z^*) = 0. \end{array} \tag{2.10}$$

The following are important special cases:

- If $l \equiv -\infty$ and $u \equiv \infty$ then MCP reduces to solving a systems of nonlinear equations; that is we try to find a point $z^* \in \mathscr{R}^q$ such that $F(z^*) = 0$.

- If $l \equiv 0$ and $u \equiv \infty$ then we obtain the standard nonlinear complementarity problem (NCP) of finding a feasible point $z^* \in \mathscr{R}^q$ for the system

$$z_i^* \geq 0,\ F_i(z^*) \geq 0,\ \text{and}\ z_i^* F_i(z^*) = 0,\ \text{for}\ i = 1,2,\dots,q. \tag{2.11}$$

- If $F(z)$ is an affine function of z in (2.11), say $F(z) = Mz + t$ for some given vector $t \in \mathscr{R}^q$ and matrix $M \in \mathscr{R}^{q\times q}$, then the problem NCP (2.11) reduces to a linear complementarity problem (LCP). The reader is referred to Cottle et al. (2009) and Murty (1988) for an extensive treatment of LCPs.

MCPs are special cases of *variational equality (VIs) problems*. We can formulate MCP (2.10) as a so-called *box constrained* variational inequality problem (2.12) since the set B is a rectangular set. z^* is a solution of MCP (2.10) if and only if it is a solution of the following VI:

$$F(z^*)^T(z-z^*) \geq 0, \ \forall z \in B. \tag{2.12}$$

The establishment of the relationship between the complementarity problem and the variational inequality problem can be found in many sources in the literature (Karamadian (1971), Harker (1993), Dirkse and Ferris (1995a), Nagurney (1999), Facchinei and Pang (2003)).

A considerable number of solution algorithms has been developed for solving mixed complementarity problems. An extensive survey on the developments in solving mixed complementarity problems is done by Ferris and Kanzow (2002). Moreover, the reader can find a comparison of the performance of the algorithms used by different MCP solvers in Billups et al. (1997)[3]. Among these solvers, PATH (Dirkse and Ferris (1995b), Ferris and Munson (2000)) has been one of the most prominent state-of-art algorithms which performs successfully on large scale mixed complementarity problems. In Chapter 3, we also formulate the equilibrium problem of generation capacity choice of the power producers in perfectly competitive electricity markets as an MCP, which we solve by utilizing the PATH solver.

Existence and uniqueness: Many of the basic theoretical results in complementarity and variational inequality problems have been already well-developed. There is a large body of literature which provide various results on the existence and uniqueness of solutions to these problems. Since it is not possible to give all these results here, we just give a flavor of the existence and uniqueness theory for complementarity problems by using the variational inequality formulation VI (2.12). The interested reader can find an extensive review of theoretical results in Harker and Pang (1990), Harker (1993), Nagurney (1999), and Facchinei and Pang (2003).

The basic result is stated in the following theorem for the existence of a solution to the VI (2.12) (hence to the MCP (2.10)).

Theorem 2.3.2. (Harker and Pang (1990)) *Let B be a nonempty, compact, and convex set and let $F(z)$ be a continuous mapping on B. Then there exists a solution to the problem VI (2.12).*

In general, the variational inequality problem may have more than one solution. If F is continuous and strictly monotone on B, then the solution of the VI (2.12) is unique, if it exists, as stated in the next theorem.

Theorem 2.3.3. (Harker and Pang (1990)) *Let B be a nonempty, closed, and convex set and F be continuous and strictly monotone on B. Then VI (2.12) has at most one solution.*

The following theorem provides a condition under which both existence and uniqueness of the solution to

[3]Also see http://www.gams.com/docs/pathvsmiles.htm for comparison between PATH and MILES.

the problem VI (2.12) is guaranteed.

Theorem 2.3.4. (Harker and Pang (1990)) *Let B be a nonempty, closed, and convex set and F be continuous and strongly monotone on B. Then there exists a unique solution z^* to VI (2.12).*

Finally, since MCPs constitute a subclass of VIs, the existence and uniqueness results for VIs also hold for MCPs.

2.3.3.1 Formulating Karush-Kuhn-Tucker (KKT) Conditions of an NLP as an MCP

Karush-Kuhn-Tucker (KKT) conditions are first order (necessary) optimality conditions of an NLP. As mentioned in Section 2.2.1, under certain conditions, KKT conditions are also sufficient optimality conditions of an NLP. One can establish the relationship between the (non)linear optimization problem and the mixed complementarity problem by formulating the KKT conditions of an NLP as an MCP (e.g., Ferris and Sinapiromsaran (1998)).

Let z in MCP (2.10) be composed of the vectors of primal variables x and dual variables λ and μ of the NLP (2.1). Then the mixed complementarity problem is to find $z^* = (x^*, \lambda^*, \mu^*) \in \mathscr{R}^q$ where $q = n+m+k$ within the lower bounds l and upper bounds u that is complementarity to the nonlinear vector function $F : \mathscr{R}^q \mapsto \mathscr{R}^q$ given in (2.13):

$$\begin{aligned} F(z^*) &:= \begin{bmatrix} \nabla_x f(x^*) - \lambda^{*T} \nabla_x g(x^*) - \mu^{*T} \nabla_x h(x^*) \\ g(x^*) \\ h(x^*) \end{bmatrix}, \\ l &:= \begin{bmatrix} 0 \\ 0 \\ -\infty \end{bmatrix}, \text{ and } u := \begin{bmatrix} \infty \\ \infty \\ \infty \end{bmatrix}, \end{aligned} \tag{2.13}$$

where f, g, and h are the functions in NLP (2.1). It can be easily seen that the first order conditions of NLP (2.1) given in (2.3) are equivalent to the MCP (2.13). Therefore, in theory, one can try to find the stationary points of an NLP problem using an MCP solver. If the KKT conditions (2.3) are necessary and sufficient for optimality then the solution of the MCP (2.13) is equivalent to the solution of the NLP (2.1).

2.3.3.2 Formulating Nash Games as MCPs

MCPs represent a broader class of problems than (N)LPs; that is MCPs are not just the optimality conditions of (N)LPs. There may not be any (non)linear optimization problem corresponding to an MCP Gabriel et al. (2012). Furthermore, using MCPs is a natural way of formulating various equilibrium problems such as Nash games. The advantage of MCPs is that we can formulate and solve the optimality conditions of several (N)LP problems, that are not separable, simultaneously. This is especially useful when we would like to find an equilibrium in a Nash game where every player's objective function depends on the strategy of other players and each player attempts to maximize (minimize) his/her utility (cost) for each fixed strategy of his/her opponents under a set of constraints (such as market clearing conditions) that tie all these optimization problems together.

Next, we elaborate on how we can formulate Nash games and compute Nash equilibria via variational inequality or mixed complementarity problem formulations. To this end, we assume that X_i in player i's problem (2.5) is represented by a set of equality and inequality constraints. Then $X_i := \{x^i \mid g_i(x^i) \geq 0,\, h_i(x^i) = 0,\, x^i \geq 0\}$, where g_i, and h_i are continuously differentiable functions. As a result, player i's problem (2.5) becomes an NLP and we can compute a Nash equilibrium in two ways (Harker (1993) and Facchinei and Pang (2003)):

- If f_i is a pseudoconvex function, g_i is a vector-valued quasiconcave function, h_i is a vector-valued affine function, and a constraint qualification holds, then by using Proposition 2.2.1 in Section 2.2.1, the solution of each player i's optimization problem (2.5) is equivalent to solving the KKT system (2.14):

$$\begin{array}{lll} 0 \leq \nabla_{x^i} f(x^{*i}, \mathbf{x}^{*-\mathbf{i}}) - \lambda^{*i^T} \nabla_{x^i} g_i(x^{*i}) - \mu^{*i^T} \nabla_{x^i} h(x^{*i}) & \perp & x^{*i} \geq 0 \\ 0 \leq g_i(x^{*i}) & \perp & \lambda_i^* \geq 0 \\ h_i(x^{*i}) = 0, & & \end{array} \tag{2.14}$$

  where λ^i and μ^i are the vector of Lagrange multipliers (dual variables) corresponding to the inequality and equality constraints of firm i, respectively.

  In order to find a Nash equilibrium, one can concentrate on solving the KKT conditions for every player in one mixed complementarity problem simultaneously (one can solve (2.14) for all i) and use an algorithm like PATH (Dirkse and Ferris (1995b), Ferris and Munson (2000)).

- Alternatively as we explain in Section 2.3.3 we can calculate a Nash equilibrium by solving a variational inequality problem. A tuple $x^* = (x^{*1}, x^{*2}, \ldots, x^{*N})$ is a Nash equilibrium if x^* is a solution of VI (2.15).

$$F(x^*)^T (x - x^*) \geq 0,\ \forall x \in X, \tag{2.15}$$

  where $F(x) = (\nabla_{x^i} f_i(x^i, \mathbf{x}^{-\mathbf{i}}))_{i=1}^N$ and $X = \prod_{i=1}^N X_i$. Since $X_i := \{x^i \mid g_i(x^i) \geq 0,\, h_i(x^i) = 0,\, x^i \geq 0\}$, VI (2.15) is equivalent to (2.14).

If X_i is a compact and convex set, then since $\nabla_{x^i} f_i(x^i, \mathbf{x^{-i}})$ is continuous with respect to x^i for each fixed $\mathbf{x^{-i}}$, the existence of Nash equilibria follows from Theorem 2.3.2. In addition if f_i is strictly convex with respect to x^i for each fixed $\mathbf{x^{-i}}$, then by Theorem 2.3.3 strict monotonicity of $F(x)$ implies that there is a unique Nash equilibrium (Harker and Pang (1990), Harker (1993), and Nagurney (1999)). Similar to the first case, we can use PATH to solve (2.15).

2.3.4 Bilevel Programming

Bilevel programming problems are hierarchical mathematical problems where the set of all decision variables is partitioned between vectors x and y to be solved at two stages. The vector x is the decision variable of the optimization problem at the first stage. Given x, the vector $y^*(x)$ is chosen as an optimal solution of the optimization problem parameterized in x at the second stage. Hence, there is an (N)LP problem at each stage and the constraints of the first-stage problem are defined in part by the second-stage problem.

To be more precise, knowing the selection x at the first stage, a decision maker selects his/her best strategy $y^*(x)$ which optimizes his/her problem (i.e., his/her cost (utility) is minimized (maximized)) at the second stage:

$$y^*(x) \in S(x) := \arg\min_{y \geq 0} \{ f(x,y) \text{ s.t. } g(x,y) \geq 0, h(x,y) = 0 \}, \tag{2.16}$$

where $f : \mathscr{R}^n \mapsto \mathscr{R}$, $g : \mathscr{R}^n \mapsto \mathscr{R}^k$, $h : \mathscr{R}^n \mapsto \mathscr{R}^m$ are differentiable functions and $y^*(x)$ is called the optimal reaction of the decision maker at the second stage on the given choice of x at the first stage. In principle, one can take $y \in \mathscr{R}^l$ in (2.16); we will consider $y \geq 0$ just to keep in line with the mathematical formulations introduced throughout this thesis.

Being aware of the second stage decision $y^*(x)$, the same or another decision maker chooses his/her best strategy $x \in \mathscr{R}^n$ which optimizes his/her problem at the first stage:

$$\min_x \{ \theta(x, y^*(x)) \text{ s.t. } x \in X, y^*(x) \in S(x) \}, \tag{2.17}$$

where $X \in \mathscr{R}^n$ is a nonempty closed set in which the constraints only involve the first stage decision x and are independent of $y^*(x)$.

The (N)LP problem (2.17) at the first stage is called the *upper level problem* whereas the (N)LP problem (2.16) at the second stage is called the *lower level problem*. The upper level problem is sometimes called *the leader's problem* and the decision maker is called *the leader* since his/her strategy at the upper level leads the decisions made at the lower level. If there is another decision maker at the lower level who is not a leader, then he/she is called *the follower* and his/her problem may also be called as *the follower's problem*. Dempe (2002) presents an extensive overview of the theory of bilevel programming, different formulations of bilevel problems as one level optimization problems, their optimality conditions, and the solution algorithms.

In a bilevel program, there may be one or several agents at both upper and lower level problems. Next, we show two special cases of bilevel program. The first case, a two-stage stochastic program, represents a problem of a single agent at both upper and lower levels whereas in the second case there is an equilibrium problem of several agents at the lower level.

2.3.4.1 Two-stage Stochastic Programs

A two-stage stochastic program, introduced in Section 2.2.2, is a special case of bilevel programs. In a general bilevel program there may be several agents at upper and lower levels whereas in a two-stage stochastic program (e.g., problem (2.4)) there is a single agent. This agent has to make a “here-and-now” decision x at the first stage before the realizations of the uncertain data $\xi(\omega)$, viewed as a random vector, are known. At the second stage, after one or many repetitive realizations of $\xi(\omega)$ becomes available, he/she obtains $y^*(\omega,x)$ and $F(\xi(\omega),x)$ by solving the lower level problem for given x. Eventually, we are interested in finding x^* which minimizes his/her expected cost, $\theta(x,y^*(x)) := E_\omega[F(\xi(\omega),x)]$, at the upper level.

As mentioned in Section 2.2.2, we formulate a two-stage stochastic program of a central decision maker in Chapter 3 and we solve the corresponding stochastic program to find the equilibrium of firms’ generation capacity choices under uncertainty of market conditions in decentralized electricity markets.

2.3.4.2 Mathematical Programs with Equilibrium Constraints (MPECs)

In game-theoretic models with multiple agents, each agent as being a leader may face a bilevel problem where the lower level problem is an equilibrium problem. Then each agent’s bilevel problem can be reformulated as a *mathematical problem with equilibrium constraints (MPEC)*. MPECs are widely used tools to reformulate the bilevel problem of a leader as a single optimization problem by replacing $S(x)$ in (2.17) with the equilibrium problem of the lower level. For instance, this suggests to replace problem (2.17) and (2.16) by

$$\begin{array}{ll} \min\limits_{x,y,\lambda,\mu} & \theta(x,y) \\ & x \in X \\ & 0 \leq \nabla_y f(x,y) - \lambda^T \nabla_y g(x,y) - \mu^T \nabla_y h(x,y) \quad \perp \quad y \geq 0 \\ & 0 \leq g(x,y) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \perp \quad \lambda \geq 0 \\ & h(x,y) = 0, \end{array} \tag{2.18}$$

where $\lambda \in \mathscr{R}^k_+$ and $\mu \in \mathscr{R}^m$ are the vector of Langrange multipliers of the inequality and equality constraints in (2.16), respectively.

Viewed as an NLP, the feasible region of an MPEC is often nonconvex and its constraints do not satisfy a classical constraint qualification such as *the linear independence constraint qualification (LICQ)* or *the Mangasarian-Fromovitz constraint qualification (MFCQ)*. Therefore, standard solution methods for nonlinear programs are prone to failure. Several specialized solution approaches for MPEC problems are proposed in the literature. Some of these are implicit programming approach, smoothing and regularization methods, penalty function approach, and sequential quadratic programming (SQP) approach. For an extensive review of these methods and several theoretical results about MPECs, we refer the reader to Luo et al. (1996) and Outrata et al. (1998).

In Chapter 4, we consider a bilevel problem for each each firm: For given generation capacities of the other firms, each firm chooses its generation capacity at the upper level and the production of all firms takes place at the lower level in future market. Furthermore, the future market conditions at the lower level are not known in advance (i.e., under demand uncertainty). We assume that each firm has market power at the first stage; that is it anticipates the impact of its capacity choice at the first stage on the market price and the production of all firms for each demand realization at the second stage. In this context, if we focus on only one firm's problem, then we can also reformulate its bilevel problem as a *mathematical problem with equilibrium constraints (MPEC)*.

Finally, in Chapter 4, we indeed consider all firms as leaders having an MPEC problem each, then one can speak of finding a Nash equilibrium to *an equilibrium problem with equilibrium constraints (EPEC)*. It is well known that EPECs are in general nonconvex problems and they often do not have solutions. Two different approaches can be found in the literature to solve EPEC problems but none of these solution methods has analytically been proved to converge to the true solution of the EPEC problem (see Gabriel et al. (2012) for details). One of these approaches is the so called ALLKKT approach, where the first order conditions of all the leaders' problems are solved simultaneously (Hu (2003), Ehrenmann (2004), and Hu and Ralph (2007)). Another approach, so-called diagonalization, is a kind of fixed-point iteration in which each leader chooses and updates his/her strategy sequentially holding the other leaders' strategies as fixed until the sequence converges (Hobbs et al. (2000), Hu and Ralph (2007), Yao et al. (2008)).

2.4 Methodology: Simulation-based Optimization under Uncertainty

Development of alternative methods to find a solution to stochastic problems is an active area of research. Among these, a class of simulation-based methods, so called *sample-path methods* (also known as *sample average approximation methods*), utilize simulation and deterministic optimization methods to solve many types of stochastic problems. In sample-path methods, a complex stochastic system is observed along a fixed sample path by using the method of common random numbers from simulation literature (e.g., Banks et al. (2010)). This way the underlying stochastic problem is converted to a deterministic problem which provides an approximate solution to the solution of the stochastic problem. The resulting deterministic problem is then solved by fast and effective deterministic solution methods available and its solution is taken as an approximate solution of the original stochastic problem.

In Chapter 3, we consider both two-stage stochastic programs and two-stage stochastic equilibrium problems in decentralized electricity markets and we utilize sample-path methods to find a solution to each of these problems. In Chapter 5, we consider a stochastic optimization problem of a manufacturing flow control system, which is modeled as a stochastic-fluid system, and find a solution to this problem by using sample-path optimization method.

This section outlines several aspects of sample-path methods for solving *stochastic optimization problems* and *stochastic equilibrium models*. We first describe how the sample-path idea is used for solving stochastic optimization problems in Section 2.4.1 and then we present, in Section 2.4.2, an extension of this method to solve stochastic equilibrium models.

2.4.1 Sample-path Optimization

In this section, we present the basic idea of *sample-path optimization method* to find a solution to a stochastic optimization problem.

Suppose we are given an extended real-valued stochastic process $\{f_M(x) \mid M \geq 1\}$ where x represents the vector of decision variables and $f_M(x)$ almost surely converges pointwise to an extended real-valued limit function $f_\infty(x)$, as $M \to \infty$. We are interested in solving a stochastic optimization problem that involves $f_\infty(x)$; however, $f_\infty(x)$ is intractable (i.e., its analytical representation is not possible) or hard to compute. For example, $f_\infty(x)$ can be a steady-state[4] performance measure of a simulation (e.g., the long run time-averaged number of customers in a system or the long run average production and/or backlog cost of the system per unit time) while $f_M(x)$ is observed for simulation length of M. $f_\infty(x)$ can also be an expectation while $f_M(x)$ is the average of M instances of the system.

Assume that $\{f_M(x) \mid M \geq 1\}$ and $f_\infty(x)$ are defined on a common probability space $(\Omega, \mathscr{F}, P)$. For each x, we denote the sample-path of the stochastic process by $\{f_M(\omega, x) \mid M \geq 1\}$ where ω represents the sample path (which is a list of random numbers that give rise to a particular simulation run), and M is the simulation length. We cannot compute $f_\infty(x)$ but we can use $f_M(\omega, x)$ for large M to approximate $f_\infty(x)$. For instance, if $f_\infty(x)$ is a steady-state performance measure, then we can simulate the corresponding system M time units and compute $f_M(\omega, x)$ which converges pointwise to $f_\infty(x)$ as $M \to \infty$. On the other hand, if $f_\infty(x)$ is an expectation, then we repeatedly simulate M instances of the system and compute $f_M(\omega, x) = \frac{1}{M}\sum_{j=1}^{M} f(\omega, x_j)$. By using the strong law of large numbers, the average of the observations, $\frac{1}{M}\sum_{j=1}^{M} f(\omega, x_j)$, converges almost surely to the expected value, $f_\infty(x)$ as $M \to \infty$. Furthermore, in most cases, we can compute derivatives or directional derivatives of the function $f_M(\omega, x)$ by utilizing well-established methods of gradient estimation (see Glassermann (1991) and L'Ecuyer (1991) for a general overview). Some commonly used gradient estimation techniques are finite-difference methods, the likelihood ratio method, and infinitesimal perturbation analysis (see Ho and Cao (1991) and Glassermann (1991)).

The method can be summarized as follows: we fix a sample path ω (by using common random numbers) and a large simulation length M (to get a good estimate of the limit function), then we obtain the deterministic function $f_M(\omega, x)$ and the corresponding problem becomes a deterministic optimization problem (e.g., NLP). By exploiting the information of (directional) derivatives of $f_M(\omega, x)$, we can solve the resulting deterministic

[4] A system is said to be in steady state provided the probability that the system in a given state is time independent.

problem with efficient deterministic (NLP) solvers available and take its solution as an approximate minimizer of the stochastic optimization problem (that involves $f_\infty(x)$). Note that when the limit function is a steady-state performance measure or an expectation, which is the case in the applications we consider in this thesis, the minimizers of the stochastic problem are independent of ω.

The theory of sample-path optimization for finding a minimizer of $f_\infty(x)$ in stochastic optimization problems with deterministic constraints is analyzed in Robinson (1996) which also contains additional references to the literature. Robinson (1996) gives a number of sufficient conditions that guarantee that if we take M large enough, each approximate minimizer of $f_M(\omega,x)$ will be close to a true minimizer of the limit function ($f_\infty(x)$). The results of Robinson (1996) guarantee that the method converges with probability one under two conditions: first, the sequence of approximating functions epiconverges to the limit function; second, the limit function almost surely has a nonempty, compact set of minimizers. These conditions hold under fairly general settings. For instance, epiconvergence is implied by pointwise convergence of convex functions ($f_M(\omega,x)$) to a convex limit ($f_\infty(x)$). Later, Gürkan et al. (1999b) establish almost-sure convergence of sample-path methods when dealing with stochastic optimization problems with stochastic constraints. In this thesis, we do not explicitly deal with theoretical issues related to convergence analysis; instead we solely focus on the operational issues and observe convergence numerically. However, for some of the stochastic optimization problems we consider in this thesis (such as two-stage stochastic programs in Chapter 3), we have convex approximating functions which converge pointwise to convex limits. Hence, sufficient conditions for convergence hold for these problems and ensure that if we take the sample size large enough, the solutions we obtain will be close to the true minimizer of these stochastic problems. The reader interested on the theory of convergence in sample-path optimization is referred to Robinson (1996) and Gürkan et al. (1999b).

Basic applications of sample-path optimization in the literature were done by Plambeck et al. (1996) and Plambeck et al. (1993) regarding the solution of simulation optimization problems with deterministic constraints. Plambeck et al. (1996) report extensive numerical experiments utilizing sample-path optimization on large scale systems (project management involving PERT networks with up to 110 stochastic arcs and cycle time optimization in unreliable tandem production lines with up to 50 machines). In turn, Gürkan (2000) focuses on optimization of buffer allocations in unreliable tandem production lines and also solves large scale systems with up to 50 machines. Both Plambeck et al. (1996) and Gürkan (2000) use IPA for gradient estimation.

Furthermore, a closely related technique centered around likelihood-ratio methods appeared in Rubinstein and Shapiro (1993) under the name of stochastic counterpart methods. The basic approach (and its variants) is also known as *sample average approximation method (SAA)* in the stochastic programming literature and is used by Shapiro and Homem-De-Mello (1998) to solve a two-stage stochastic program with recourse structure.

In this thesis, we utilize sample-path optimization or sample average approximation methods to find a solution to the stochastic optimization problems we introduce in Chapters 3 and 5. It is worth to note that the types of stochastic optimization problems we tackle in these chapters allow us to use sample-path methods in order to find the optimal decisions. In Chapter 3, we deal with finding the optimal investment decisions in a two-stage stochastic program for which the theory of sample-path method and the statistical inference for its convergence are quite well-developed (Shapiro and Homem-De-Mello (1998), Linderoth et al. (2006)). However, the sample-path method which shows a considerable promise for solving two-stage stochastic programs may be practically inapplicable for solving multistage stochastic programs with a large (e.g., greater than 4) number of stages (see Shapiro (2006) and Shapiro et al. (2009) for the computational complexity of solving multistage stochastic problems with sample-path methods). In Chapter 5, we consider a control problem of a stochastic manufacturing system and we are concerned with optimizing the production parameters (i.e., target inventory levels) of a plausible class of policies proposed for such type of problems in the literature. However, it is not much known whether sample-path methods would work to find optimal policies of a stochastic optimal control problem, in particular with infinite horizon. Next, we briefly describe the implementation of sample-path methods we use in Chapters 3 and 5. More details of these methods can be found in the corresponding chapters.

In Chapter 3, we formulate a two-stage stochastic program of a central decision maker under different electricity market designs, where the generation capacities are chosen at the first stage under uncertainty of future market conditions and power generation takes place in the future spot market at the second stage. The two-stage stochastic program involves minimizing the total expected cost of the electricity system as the limit function. We cannot observe the expected cost which is the objective function in the first-stage problem; however we can approximate it by a sample-average function using the sample $\{\omega_m\}_{m=1}^{M}$. Thus, we utilize sample-path optimization or sample average approximation method in stochastic programming literature to find its approximate solution (which we also show to be the equilibrium of the perfectly competitive decentralized electricity markets). The approximate problem is in general an NLP and we can solve it by using a standard NLP solver in two ways:

- One way is solving a single NLP which includes both first stage variables and all the second stage variables for the sample $\{\omega_m\}_{m=1}^{M}$. However, the size of the NLP may get very large for large sample sizes which increases the computational time. Thus, we use in Chapter 3 an alternative approach as explained below.

- Alternatively, we can solve the first and second stage problems iteratively by using an (N)LP solver at each stage as illustrated in Figure 2.2 (a). We model the approximate problem at the first stage in which we define the sample average cost function as an external function. In order to solve the approximate problem at the first stage, we solve the second-stage problem M consecutive times at any first stage decision point x that the NLP solver (at the first stage) would like to evaluate its optimality. For given x, we obtain the corresponding primal-dual solutions from the second stage (N)LP problem and we use these primal-dual solutions in the first stage problem to calculate the value and the (directional) derivatives of the sample average cost function. The iterations continue until the optimality conditions of both the NLP problem at the first stage and the (N)LP problem at the second stage are satisfied at a decision point x^*. Thus, x^* is taken as the optimal generation capacity investments in the electricity market and the primal-dual solutions of the second stage problem at x^*, for each $m \in M$, are taken as the optimal spot market outcomes.

Furthermore, both the two-stage stochastic program and its approximate problem given in Chapter 3 are convex problems with nonempty solution sets. Hence as mentioned above, this guarantees convergence of the sample-path method by using the results of Robinson (1996).

In Chapter 5, we consider a manufacturing flow control problem that is modeled as a stochastic fluid system. We are interested in finding the optimal threshold (target inventory) levels of a plausible class of control policies, so-called hedging policies, minimizing the long run time-averaged production and backorder cost of the manufacturing system. The system is incurred under different production and demand states that change according to some external stochastic process. The long run average cost depends on the production rate, which is adjusted in each state depending on the threshold (hedging) level.

In order to solve this problem, we use a similar approach to Gürkan (2000): We develop a generalized semi-Markov process (GSMP) representation of the system. A GSMP can simply be thought as a mathematical framework that models the evolution of a discrete-event simulation (Glynn (1989) and Shedler (1993)). By utilizing the GSMP representation together with IPA for gradient estimation, we can derive recursive expressions to calculate the value and the exact derivatives of the approximating average cost function in a single simulation run of M time units. Thus, we fix a sample-path by using common random numbers and solve the approximating NLP problem for a large M by using a state-of-the-art NLP solver. The illustration of the corresponding sample-path method is given in Figure 2.2 (b). In the NLP model, we define the approximating average cost function as an external function. We use an iterative procedure in which the NLP solver calls the simulation model (GSMP) at any decision point x that it would like to evaluate its optimality. The GSMP model enables us to obtain the value and the (directional) derivatives of the approximating average cost function at a single simulation run at each iteration. The iterations continue until the optimality conditions of the NLP problem at a decision point x^* are satisfied. Thus, x^* is taken as the optimal threshold levels of the system. Furthermore, we also include stochastic constraints in the manufacturing flow control problem and use sample-path method in a similar way to solve the approximating problem. In the modified stochastic problem, we constrain the long run proportion of time that the system is in backlog state below a pre-specified level.

2.4.2 Solving Stochastic Equilibrium Models: Stochastic MPCs/VIs

In this section we present an extension of the sample-path optimization method discussed in Section 2.4.1 to solve stochastic MCPs or variational inequality problems. Since stochastic MCPs are a subclass of stochastic VIs, we will explain the sample-path method for stochastic VIs. Consider the following VI problem in which we are interested in finding a point $z^* \in B$, if it exists, satisfying

$$F_\infty(z^*)^T(z-z^*) \geq 0, \ \forall z \in B, \tag{2.19}$$

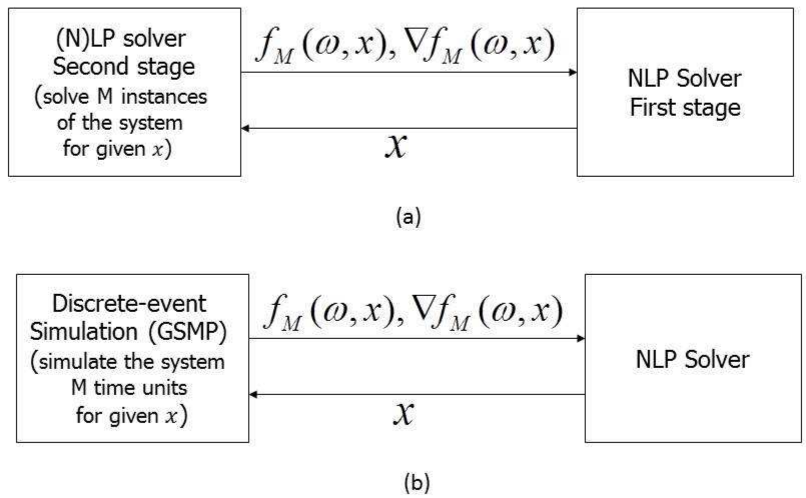


Figure 2.2: Illustration of the implementation of sample-path methods in (a) Chapter 3 and (b) Chapter 5, respectively

where $F_\infty(z^*)$ are vector-valued functions involving expectations or steady-state performance measures in a stochastic system and B is polyhedral convex set.

As an extension of sample-path optimization, the sample-path method for solving stochastic VIs inherits its important features, thus it is very similar to its optimization counter part. Let $F_M(\omega, z)$, observed by a simulation length of M, be approximating functions of $F_\infty(z)$. As in the sample-path optimization method described in Section 2.4.1, we fix a large M to get a good estimate of $F_\infty(z)$ and a sample point ω using the method of common random numbers from simulation literature. For fixed M and ω, $F_M(\omega, z)$ is a deterministic function of z. Hence, we solve the deterministic variational inequality defined by B and $F_M(\omega, z)$ and take its solution as an approximate solution of the problem VI (2.19). By utilizing gradient estimation methods, one can use efficient deterministic algorithms (i.e., PATH by Dirkse and Ferris (1995b) and Ferris and Munson (2000)) to solve the approximating variational inequality problem, which is a very important feature since unlike stochastic optimization problems there are currently no other methods to solve stochastic variational inequalities.

In Gürkan et al. (1996) and Gürkan et al. (1999a), the basic idea of using sample-path information is extended to solving stochastic variational inequalities and equilibrium problems. For an extensive analysis of sample-path method for stochastic variational inequalities and the sufficient conditions for convergence of the method, the reader is referred to Gürkan et al. (1999a).

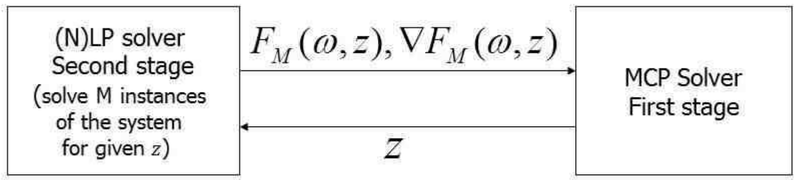


Figure 2.3: Illustration of the implementation of sample-path method to solve stochastic MCPs in Chapter 3

In Chapter 3, we also use sample-path methods for finding an approximate of the solution of stochastic mixed complementarity problems. As briefly mentioned in Section 1.2.1.1, we formulate, in Chapter 3, stochastic equilibrium problems for different electricity market designs under perfect competition. These equilibrium problems are formulated as stochastic MCPs consisting of the vector-valued expected performance functions of the first stage generation capacity decisions under uncertainty of spot market outcomes at the second stage. We can approximate the corresponding stochastic MCPs by using the sample $\{\omega_m\}_{m=1}^M$. Thus, we can find an estimate of the solution to these stochastic MCPs by solving the approximating MCP problems. In particular, for one case of operating-reserve pricing scheme we need to solve the equilibrium problem as a stochastic MCP since we cannot cast it as a single stochastic optimization problem. Similar to the implementation of sample-path optimization methods described in Section 2.4.1, we can solve the approximating MCP in two ways:

- One way is solving a single MCP which includes both first stage variables and all the second stage variables for the sample $\{\omega_m\}_{m=1}^M$. However, the size of the MCP may get very large for large sample sizes, which increases the computational time as we also observe in our numerical experiments.

- Alternatively, we can decompose the first and second stage variables and solve both problems iteratively. For the equilibrium problems formulated in Chapter 3, we have an MCP problem at the first stage whose second stage decision variables are solutions to an (N)LP problem (see Chapter 3 for details). Thus, we solve the first stage problem by an MCP solver (e.g., PATH) and the second stage problem by an (N)LP solver iteratively as illustrated in Figure 2.3. We model the approximate problem at the first stage as an MCP in which we define the approximating functions $F_M(\omega, z)$ as external functions of z. In order to solve the approximate problem at the first stage, we solve the second-stage problem M consecutive times at any first stage decision point z that the MCP solver (at the first stage) would like to explore. For given z, we obtain the corresponding primal-dual solutions from the second stage (N)LP problem and we use these primal-dual solutions in the first stage problem to calculate the value and the (directional) derivatives of the the approximating functions with respect to z. The iterations continue until the MCP solver finds a solution z^* (i.e., generation capacities) which is taken as the first stage equilibrium in the electricity market. The primal-dual solutions of the second stage (N)LP problem at z^*, for each $m \in M$, are taken as the spot market outcomes at equilibrium.

Finally, as in the application of sample-path optimization method, we do not deal with convergence analysis theoretically; instead we observe convergence numerically.

Chapter 3

Generation Capacity Investments in Electricity Markets: Perfect Competition

Abstract

In competitive electricity markets, markets designs based on power exchanges where supply bidding (barring demand-side bidding) is at the sole short run marginal cost may not guarantee resource adequacy. As alternative ways to remedy the resource adequacy problem, we focus on three different market designs in detail when demand is inelastic, namely an energy-only market with VOLL pricing (or a price cap), an additional capacity market, and operating-reserve pricing. We also discuss demand-side bidding (i.e., a price responsive demand) which can be seen as a categorically different alternative to remedy the resource adequacy problem. We consider a perfectly competitive market consisting of three types of agents: generators, a transmission system operator, and consumers; all agents are assumed to have no market power. For each market design, we model and analyze capacity investment choices of firms using a two-stage game where generation capacities are installed in the first stage and generation takes place in future spot markets at the second stage. When future spot market conditions are assumed to be known *a priori* (i.e., deterministic demand case), we show that all of these two-stage models with different market mechanisms, except operating-reserve pricing, can be cast as single optimization problems. When future spot market conditions are not known in advance (i.e., under demand uncertainty), we essentially have a two-stage stochastic game. Interestingly, an equilibrium point of this stochastic game can be found by solving a two-stage stochastic program, in case of all of the market mechanisms except operating-reserve pricing. In case of operating-reserve pricing, while the formulation of an equivalent deterministic or stochastic optimization problem is possible when operating-reserves are based on observed demand, this simplicity is lost when operating-reserves are based on installed capacities. We generalize these results for other uncertain parameters in spot markets such as fuel costs and transmission capacities. Finally, we illustrate how all these models can be numerically tackled and present numerical experiments. In our numerical experiments, we observe that uncertainty of demand leads to higher total generation capacity expansion and a broader mix of technologies compared to the investment decisions assuming average demand levels. Furthermore for the same VOLL (or price cap) level and under the assumptions of random demand with finite support and no forced outages, energy-only markets with VOLL pricing tend to lead to total generation capacity below

the peak load with a certain probability whereas energy markets with a forward capacity market or operating-reserve pricing result in higher investments. Finally, the regulator decisions (e.g., reserve capacity target) in capacity markets and operating-reserve pricing can be chosen in such a way that results in very similar investment levels and fuel mix of generation capacities in both market designs.

3.1 Introduction

In the days of regulated monopolies, there was no issue of generation resource adequacy. Companies were obligated to serve the demand and had to invest accordingly; an optimization model was used to compute the expansion of generation capacity decisions that would satisfy demand at minimal investment and operations costs (subject to reliability constraints that we will not discuss here). In compensation for this obligation, electricity prices were regulated (often at average cost) in a way that guaranteed that the company could pay for its expenses (including reimbursement of long term debt) and make a reasonable profit on equity. The theory of peak load pricing for non-storable commodities, such as electricity, is the economic counterpart of these computational models; it can be seen as an economic interpretation of the capacity expansion model in terms of electricity prices that induce efficient investments and operations. Of particular importance, the theory of peak load pricing explains that the price of electricity in the highest demand period must embed a particular (peak load) component to induce an efficient capacity mix. Both the capacity expansion models and the theory of peak load pricing can be traced back to work conducted in Électricité de France in the fifties and sixties (see the collection of early papers treating both subjects in Morlat and Bessière (1971)).

Capacity expansion models were extensively developed during the regulatory periods before loosing some of their appeal after restructuring. After restructuring, generation and investments became the responsibility of companies who had to make a profit on the electricity market. This gave rise to the question whether energy-only electricity markets would provide incentives for adequate investments and thereby maintain security of supply. This discussion focuses especially on those power plants which will only be needed to meet demand at peak hours and therefore have to earn sufficient revenues in those hours to cover their investment costs. The theory of peak load pricing, which was initially developed for the regulated monopoly, was later proved equally relevant to perfectly competitive markets (in case of elastic demand, see Crew et al. (1995) for a survey). This theory has become crucial today to explain why competitive electricity markets may not spontaneously provide the right incentive to invest in generation capacity for peak load and to suggest remedies to this market failure.

In this chapter, we consider three variations of competitive electricity market designs known as energy-only market with VOLL pricing (or a price cap), a forward capacity market, and operating-reserve pricing as possible remedies to a market failure of insufficient generation capacity investment. We also discuss demand-side bidding which is strictly speaking not a remedy to a market failure but an alternative way to remove this market failure. We formulate generation capacity investment decisions in these electricity market designs as two-stage equilibrium problems, which is a natural way of modelling problems with multiple decision makers in a competitive environment. We show that most of these equilibrium models can be cast in mathematical programming formulations that are not too far from the early capacity expansion models. Establishing the exact relations between the two-stage equilibrium problems and the early capacity expansion models under these market designs is one of the main objectives of this chapter. We continue this introduction by formalizing the question of resource adequacy in competitive electricity markets. We do so by referring to the early capacity expansion models (in the most simplified setting) and to the interpretation of their dual solution in peak load pricing terms.

Consider the following simple generation capacity expansion model of a regulated monopoly where supply and demand are located at a single node. There is a finite set of plant types K and a finite set of time segments Ω (these can also be interpreted as states of the world, as we later do) each occurring with some duration $\pi(\omega)$ (that we later interpret as probabilities). $d(\omega)$ is the demand in time segment ω; κ_k and c_k are the unit capacity and unit generation cost of plant k; x_k is the capacity of plant type k and $y_k(\omega)$ is the generation of this plant in time segment ω. Assuming the monopoly firm is regulated in a way that motivates cost minimization, the capacity expansion model, together with its dual variables, $\beta(\omega)$[1] and $p(\omega)$[1], is stated as:

$$
\begin{aligned}
&\min \sum_k \kappa_k x_k + \sum_\omega \pi(\omega) \sum_k c_k y_k(\omega) \\
&\text{s.t. } x_k - y_k(\omega) \geq 0 \qquad\qquad\qquad \pi(\omega)\beta_k(\omega) \quad \forall \omega \quad \forall k \\
&\qquad \sum_k y_k(\omega) - d(\omega) \geq 0 \qquad\qquad \pi(\omega)p(\omega) \quad \forall \omega \\
&\qquad y_k(\omega) \geq 0 \qquad x_k \geq 0 \quad \forall \omega \quad \forall k.
\end{aligned}
\tag{3.1}
$$

This model allows the monopoly firm to determine an optimal investment portfolio of generation capacity mix for satisfying various demand levels[2]. The closely related producer and consumer surplus maximization problem has a richer economic content. Let $P(\omega, d)$ be the inverse demand function of the market where the price is given as a function of quantity supplied in time segment $\omega \in \Omega$. The producer and consumer surplus maximization problem is written as:

$$
\begin{aligned}
&\max \sum_\omega \pi(\omega) \left[\int_0^{d(\omega)} P(\omega, \xi) d\xi - \sum_k c_k y_k(\omega) \right] \quad - \sum_k \kappa_k x_k \\
&\text{s.t. } x_k - y_k(\omega) \geq 0 \qquad\qquad\qquad\qquad \pi(\omega)\beta_k(\omega) \quad \forall \omega \quad \forall k \\
&\qquad \sum_k y_k(\omega) - d(\omega) \geq 0 \qquad\qquad\qquad \pi(\omega)p(\omega) \quad \forall \omega \\
&\qquad y_k(\omega) \geq 0 \qquad x_k \geq 0 \quad \forall \omega \quad \forall k,
\end{aligned}
\tag{3.2}
$$

where $\int_0^{d(\omega)} P(\omega, \xi) d\xi$ is known as the consumers' willingness to pay at $d(\omega)$. The KKT conditions of problem (3.2) provide the necessary relations to explain the resource adequacy problem. Specifically the KKT

[1]Note that we multiply these dual variables with their probabilities in the models (3.1) and (3.2) for the purpose of scaling.

[2]In the formulation (3.1), the demand constraint is assumed to be always binding if $c_k > 0, \forall k$. In more general market models with transmission or unit commitment constraints, however, the optimal solution might not be binding since technical and economic reasons may drive generators to run even when power supply exceeds the demand. In these situations, generators may seek to maintain output by offering to pay wholesale buyers to take their electricity. This could yield negative price in some locations or periods.

condition associated with a positive generation variable $y_k(\omega)$ is stated as:

$$c_k + \beta_k(\omega) = P(\omega, d(\omega)) = p(\omega), \quad \forall \omega. \tag{3.3}$$

Alternatively, the KKT condition associated with a positive investment variable x_k is stated as:

$$\sum_{\omega} \pi(\omega)\beta_k(\omega) = \kappa_k. \tag{3.4}$$

The economic interpretation of (3.3) is that a plant of type k which is operating in time segment ω generates a (scarcity or capacity) rent $\beta_k(\omega)$ in that time segment; this rent is equal to the difference between the electricity price $p(\omega)$ in that time segment and the plant's fuel cost c_k. The economic interpretation of (3.4) is that one invests in a plant of type k when the duration (π) weighted sum of the rents $\beta_k(\omega)$ over all time segments ω is equal to the investment cost κ_k. Departing from this pricing scheme induces inefficiencies, either at the consumption or generation side. The question of resource adequacy is whether restructured electricity markets lead to electricity prices that satisfy these relations. If not, there is a resource adequacy problem.

It is now recognized that the original restructured electricity markets do not spontaneously satisfy relations (3.3) and (3.4). Because electricity is not storable, the market (barring demand-side bidding) clears in the short run when demand is inelastic. The market therefore does not ensure (3.3), because it does not face a downward sloping demand curve in the short run. Instead, the organization of the market sets the price at the bid of the last plant selected to satisfy the demand, which used to be a widely used pricing scheme when liberalisation was introduced in many countries such as in Europe and in the US. Barring market power and demand-side bidding, this price is equal to the fuel cost of the last selected power plant in time segment ω. This implies that the most expensive unit in operations over the different time segments ω (the peak plant) does not make any margin and hence (3.3) is not satisfied for that unit. This can be interpreted as follows: the electricity price in the peak does not incorporate the necessary component, called as "scarcity rent", that pays for the capacity at peak demand. The peak plant therefore appears with a zero margin in (3.4) which is thus also never satisfied for that equipment. This missing margin is now commonly referred to as the missing money. Stoft (2002) was among the first ones to analyze investments by invoking insufficient payments for capacity. He explained how, barring well developed demand-side bidding, pools and power exchanges prevent prices on the market to reflect scarcity in generation capacity (see also Cramton and Stoft (2005, 2006)). Both Hogan (2005) and Joskow (2007, 2008) also offer enlightening and in depth analysis of the missing money. Oren (2007) provides a comprehensive description of the different techniques aimed at restoring resource adequacy; he also discusses various implementations and gives an extensive list of references.

Guaranteeing resource adequacy therefore requires eliminating the missing money by creating enough capacity rent $\beta_k(\omega)$ to cover the capital costs of an efficient generation system, including the capacity cost of the peak plant. This in turn requires changing the electricity pricing mechanism. In an energy-only system, the idea is to price electricity at a high value that is supposed to reflect the value of lost load (VOLL), known as VOLL pricing, when demand is curtailed. In perfectly competitive markets, Stoft (2002) shows that VOLL pricing results in optimal generation capacity investments. However, in reality VOLL is difficult to estimate and therefore a price cap (which is in general lower than VOLL) is used. If the price cap induced by the regulator is not high enough, this will constrain prices from rising up to their competitive levels at peak hours, yielding underinvestment in generation capacity (e.g., see Joskow (2008)). An alternative solution for avoiding market failure is to implement a capacity market where the regulator imposes some capacity target in line with historical data and expected demand and the firms contributing to the sufficient investment level receive numerations accordingly. In order to increase the robustness against market power, many variants of capacity markets have been proposed regarding the implementation of the market design and the treatment of demand response (e.g., Cramton and Stoft (2005), Joskow (2008), Hobbs et al. (2007), Cramton and Ockenfels (2011)). Here we assume a forward capacity market where capacity is auctioned before the investment decision is made and the resulting capacity payment is certain for the lifetime of the plant. A more sophisticated alternative remedy is to apply some form of reliability or operating-reserve pricing so that electricity price increases when the reserve margin decreases (Stoft (2002), Hogan (2005), Hogan (2009)). Lastly, as a categorically different alternative, ensuring a price responsive demand in the short-run as well as in the long-run may remedy the resource adequacy issue.

In recent years, more and more countries have implemented (e.g., US states) or are planning to implement a variety of these market mechanisms aimed at stimulating investments in new generation capacity, such as scarcity pricing when capacity is inadequate or capacity payments via additional capacity markets next to the electricity market. For policy analysis in real world problems, practical tools are needed to gain insights into the social implications of a market design or a policy target (see e.g., Schroeder (2012) and Allcott (2012) for examples of real world applications). Thus, we concentrate on these four mechanisms (including demand-side bidding briefly) in our analysis with the following contributions. We first expand on existing short run equilibrium models of restructured systems to include generation capacity decisions under these resource adequacy mechanisms and uncertainty about future electricity market conditions. The natural approach is to resort to complementarity formulations as this mathematical programming paradigm has been extensively used to model restructured electricity systems. We then assess the extent to which these models can be restated as optimization problems as solvers for optimization problems are now numerous and quite powerful. Furthermore, we provide insights about to what extent the investment incentives are affected by these different mechanisms under demand uncertainty. To this end, we illustrate how all these models can be numerically tackled and present some numerical experiments.

We assume price-taking firms and hence exclude market power. Even if real markets may depart from perfect competition, perfect competition models provide an essential benchmark for imperfect competition models. Moreover, real markets may suffer from inefficiencies as a result of regulatory intervention or market design. Utilizing perfect competition models still allows policy makers to gain insights into the social implications of a market design or a policy target (e.g., Allcott (2012)). In addition, there are computational reasons. Multi-stage imperfect competition models are difficult if not impossible to solve. This is already true for two-stage investment and operation models that are EPEC (equilibrium problem subject to equilibrium constraints) when involving market power (see Ralph and Smeers (2006), Hu and Ralph (2007) for more details on these and related complications). More general models involving a sequence of cycle of investment and operations are at this stage computationally unexplored. Last, the experience of reformed markets indeed shows that market power mitigation instruments are effective and that properly designed reformed markets function competitively and hence can be modeled under the perfect competition assumption. For example, the market monitoring results of PJM (2012) and CASIO (2012) indicate that market prices are at or near competitive levels most of the time in US states. Furthermore, the European Union and its Member States are underway to move towards a fully integrated European electricity market by 2014 with the aim to increase competition and maximize the economic welfare of all players. In some of the regions (e.g., Germany-Belgium-France-The Netherlands) where the integration has already taken place for some time, significant price convergence is observed between the countries in most of the hours, which is a good indicator for competitiveness (see DG ENERGY (2012)). To sum up, in this chapter we will deal with a computable representation of the incentives to invest in power markets functioning under perfect competition.

In reality, generation capacity expansion is a multi-period process. The market induces the creation of new capacities and the retirement of old ones. A full version of the capacity expansion model therefore involves a sequence of successive cycles of investment and operations (e.g., Schroeder (2012)). We limit our analysis to simplified models that represent a single cycle of investment and operation: investment takes place in the first stage at some investment costs; the market operates in the second stage with generators collecting sales revenues and incurring fuel costs. This restriction is made for the sake of the presentation. In contrast with multi-period imperfect competition models, it is perfectly possible to implement the mechanisms considered here in a multi-period context since convexity is in general preserved under perfect competition.

Although we focus mainly on electricity demand being uncertain and/or fluctuating with a very low elasticity, our methodology and results summarized below can be easily generalized to spot markets with other uncertainties (i.e., fuel costs, transmission capacities, wind generation etc.). For given wind capacity and volatile wind generation, one can substitute demand with residual demand levels (i.e., demand minus wind). Under uncertainty about future electricity market conditions, real world problems including generation capacity investment decisions lead to stochastic equilibrium problems (e.g., Schroeder (2012) and Allcott (2012)) that are large scale and computationally more complex to solve than a stochastic optimization problem. Equilibrium problems are indeed broader than optimization problems which is addressed in detail by Gabriel et al. (2012). In this thesis, we not only emphasize the link between optimization and equilibrium problems but also show that, regarding the problem of generation capacity investments in perfectly competitive electricity markets, most of the formulations of stochastic equilibrium problems can be cast as two-stage stochastic programs.

We consider a perfectly competitive market consisting of three types of agents namely generators, a transmission system operator (TSO), and consumers; all agents are price takers. Generators are assumed to be risk neutral and maximize their expected profits. The transmission system operator sells transmission services in order to maximize the value of its infrastructure and consumers are simply represented by an inelastic demand in most of the chapter, except in Section 3.6 where we consider a price responsive demand. It may also be of interest to include risk averseness of generators by using "coherent risk measures" (e.g., Ehrenmann and Smeers (2011) and Ralph and Smeers (2011)). The resulting problems are still stochastic equilibrium problems which are somewhat modified versions of the stochastic equilibrium problems presented in this chapter. When the markets are "perfectly competitive" and "complete", the formulation of an equivalent two-stage stochastic program is still possible as shown by Ralph and Smeers (2011). The main contributions of this chapter can be summarized as follows:

- We expand the existing short-run models of restructured electricity system to include both uncertainty of spot market conditions (i.e., demand uncertainty) and resource adequacy mechanisms, such as VOLL pricing, capacity market, and operating-reserve pricing, and also demand-side bidding as an alternative.

- In perfect competition, we show that most of the formulations of these "equilibrium" models are in fact equivalent to or can be cast as optimization problems. In particular, in the stochastic setting this result indicates the prevalence of two-stage stochastic programming for providing solutions to stochastic equilibrium models. We believe that this result is helpful for making an economic assessment of the investment incentives in new generation capacity in real world systems for the following reasons: in Sections 2.2.2 and 3.9, we explain how we can solve a two-stage stochastic program as a nonlinear or linear optimization problem. Due to the availability of powerful and efficient nonlinear programming solvers and decomposition methods for two-stage stochastic programs, solving a two-stage stochastic program is computationally much faster than solving a stochastic equilibrium model for large scale systems, which we also observe in our numerical experiments. In only one case of operating-reserve pricing, we obtain a complementarity problem that is not equivalent to an optimization problem.

- In perfect competition, we show that single and two-stage representation of these "equilibrium" models are equivalent. In other words, the open loop equilibria and the closed loop equilibria of these models coincide.

- We use sample-path methods and provide detailed algorithmic approaches for numerically tackling all these models, which allows solving these computationally complex stochastic problems by utilizing deterministic off-the-shelf solvers. We also illustrate that these algorithmic approaches help further decrease the computational time compared to solving the two-stage equilibrium problem as a potentially very large MCP (Mixed Complementarity Problem) including all the first and second stage decision variables.

- Through numerical experiments, we gain insights on the impact of demand uncertainty and to what extend these different mechanisms can remedy the resource adequacy issue. In particular, we first find that uncertainty of demand leads to higher total generation capacity expansion and a broader mix of technologies compared to the investment decisions assuming average demand levels. Furthermore for the same VOLL (or price cap) level, energy-only markets with VOLL pricing tend to lead to total generation capacity below the peak load with a certain probability whereas energy markets with a forward capacity market or operating-reserve pricing result in higher investments assuming random demand with finite support and no forced outages[3]. Last, the regulator decisions (e.g., reserve capacity target) in capacity markets and operating-reserve pricing can be chosen in such a way that results in very similar investment levels and fuel mix of generation capacities in both market designs.

The rest of the chapter is organized as follows. In Section 3.2, we give the set up and notation used throughout the chapter. In Section 3.3, we first introduce the deterministic investment and operations model (in the context of a two-stage equilibrium problem under perfect competition) and present it in two different formulations. One is a single-stage (open loop) version of the model where generators simultaneously invest and decide operations knowing future market prices. The other formulation is a two-stage (closed loop) model in which investment and operation decisions are made sequentially. Generators operate the capacities inherited from the first stage to maximize their profits. This market operation results in marginal values of plants that generators take into account in the first stage in order to decide on their investments. The distinction between open and closed loop models is important when there is market power. We show that this distinction is irrelevant here: both models are equivalent and can be reformulated as a single optimization problem of the standard capacity expansion type. Although Section 3.3 begins with a deterministic model, the rest of the chapter elaborates on stochastic models involving demand that is unknown at the time of investment. In Section 3.3.2 we extend the formulation to a stochastic energy-only equilibrium model and again find that it is equivalent to a stochastic capacity expansion model, which is a two-stage stochastic program. In Section 3.4, we take up the capacity market formulation, which we find again equivalent to a convex stochastic programming problem. In Section 3.5, we consider the more novel question of operating-reserve pricing for which we give two formulations that differ by the computation of the operating-reserves. One formulation refers the reserve to observed demand; the other refers it to the total capacity. The former one turns out to be a convex stochastic optimization problem, but the latter is not. In Section 3.6, we give a brief discussion of demand-side bidding which we treat by assuming price responsive demand; this can be thought of a completely different way of addressing resource adequacy issue. Section 3.7 outlines how all our results can be generalized for other uncertain elements in spot markets, such as unit generation costs and transmission capacities. We discuss various algorithmic approaches for handling these models numerically in Section 3.8. We report numerical results in Section 3.9 and provide insights to what extent the remedies to the missing money, discussed in Sections 3.3, 3.4, and 3.5, incite the generators to invest in generation capacity. Conclusions terminate the chapter. Finally, Appendices 3.A contains the proofs of some theorems, lemmas, and propositions.

3.2 Set-up and Notation

We consider a market with a regulator and three types of agents; namely generators, a transmission system operator (a TSO), and consumers. Generators and the TSO are price takers and maximize profits at given

[3]Note that under different assumptions (e.g., forced outages, demand distribution without a finite support), VOLL pricing (with high values of VOLL) may also result in similar investment levels as the other market designs (see the result of Hobbs et al. (2001)).

prices; consumers are represented by an inelastic demand. Generators and consumers are spatially distributed in an electricity transmission network which is operated by the TSO. Generation and transmission of electricity take place in the spot market and the locational marginal pricing is assumed to clear the spot market.

The regulator intervenes to remedy the lack of incentive to invest; his/her role differs depending on the market design. In an energy-only electricity market, he/she sets the price of the unserved energy (VOLL) or the price cap in case of demand curtailment. In an electricity market with a forward capacity market, he/she sets the capacity target to guarantee resource adequacy and rewards the firms who contribute to the sufficient investments to reach the target. Finally, in an electricity market with operating-reserve pricing, he/she sets the price of the operating-reserve and provides the firms with additional payments whenever the systems total reserve is scarce.

We consider agents interacting in a two-stage set-up: generators invest in their generation capacities in the first stage and the generation is dispatched in the second stage where the spot market clears to satisfy demand under transmission limitations. Under all market designs, the demand is first assumed to be constant during the whole year. Then we consider a random demand which varies over a year and extend our analysis under uncertainty of demand. The following notation would apply in a purely deterministic world:

Sets

N	:	set of all demand nodes
G	:	set of all firms
I_g	:	set of supply nodes of firm $g \in G$
I	:	set of all supply nodes ($I := \cup_g I_g$)
K_g	:	set of plant types of firm $g \in G$
L	:	set of electricity transmission lines in the network

Parameters

c_{ik}^{g}	:	unit generation cost of plant type $k \in K_g$ owned by firm $g \in G$ at supply node $i \in I_g$
κ_k	:	unit capacity cost of plant type $k \in K_g$
d_n	:	demand at node $n \in N$
$PTDF_{l,j}$	:	power transmitted through line $l \in L$ due to one unit of power injection from node $j \in \{N \cup I\}$ to an arbitrary hub[4] node
h_l	:	capacity limit of line $l \in L$
$VOLL$	:	the value of unserved energy or lost load

[4] $PTDF$ is calculated based on a hub node in $n \in N$ in a standard DC load flow model. The choice of hub node is arbitrary. That is, the flows resulting from a power injection at one node and an equal withdrawal at another do not depend on the location of the hub.

Variables:

Second Stage:

y_{ik}^{g}	:	quantity of power generated by plant type $k \in K_g$ of firm $g \in G$ at supply node $i \in I_g$
f_j	:	net power flow dispatched by TSO from node $j \in \{N \cup I\}$
δ_n	:	unserved (curtailed) energy at node $n \in N$
p_j	:	locational market price (nodal price) at node $j \in \{N \cup I\}$ which corresponds to shadow price of market clearing constraint

First Stage:

x_{ik}^{g}	:	capacity of plant type $k \in K_g$ owned by firm $g \in G$ at node $i \in I_g$.

3.3 Energy-only Market

We start our analysis with an energy-only electricity market. In an energy-only market with inelastic (exogenous) demand, the price of electricity is set by the market at the bid of the most expensive plant generating unless the demand is curtailed. When demand is curtailed, the price is capped by the regulator at the value of loss load (VOLL) or a price cap. During the hours of curtailment, the peak plant obtains extra margin to compensate its missing money for the whole year. The demand is first assumed to be constant during the whole year in Section 3.3.1. In Section 3.3.2, we consider a random demand which varies over a year.

3.3.1 Two-stage Equilibrium Model with Constant Exogenous Demand

In Section 3.3.1.1, we give the exact formulation of the interactions between the agents in an energy-only market at both stages when demand is fixed and we show some characteristics of both the short run and the long run perfect competition equilibria. By using these characteristics, we show in Section 3.3.1.2 that solving a single optimization problem where all the generation capacities are determined by a central decision maker finds a perfect competition equilibrium of the two-stage game introduced in Section 3.3.1.1.

3.3.1.1 The Perfect Competition Equilibrium as Mixed Complementarity Problem

We next formulate each agent's problem in the two-stage game where firms give their investment decisions, x, simultaneously at the first stage and they decide on their optimal generation levels, y, in the spot market at the second stage. Note that demand is exogenous and known to all firms who are price takers at both stages.

Second Stage: At the second stage, each firm $g \in G$ maximizes its short term profit from the spot market by optimization problem (3.5):

$$
\begin{aligned}
\Pi_g^*(x^g) := \max_{y^g} \quad & \sum_{i\in I_g}\sum_{k\in K_g}(p_i - c_{ik}^g)y_{ik}^g \\
s.t. \quad & \\
Cons\_g(x^g): \quad & \boxed{\begin{array}{lll} y_{ik}^g \le x_{ik}^g & (\beta_{ik}^g) & \forall i\in I_g, k\in K_g \\ y_{ik}^g \ge 0 & & \forall i\in I_g, k\in K_g, \end{array}}
\end{aligned}
\tag{3.5}
$$

where β^g is the vector of Lagrange multipliers associated with the capacity constraints ($y^g \le x^g$) and is referred as capacity/scarcity rent. Note that the nodal prices, p, enter as parameters to firms' second stage problems. Since all firms are price-takers under perfect competition, they act as if they cannot affect the value of p. As an important consequence, the nodal prices will also be taken as parameters in firms' first stage problems given in (3.10).

For a given set of generation decisions $\{y^g\}_{g\in G}$ of the firms and nodal prices $\{p_j\}_{j\in\{N\cup I\}}$, if there exists a price difference between any two nodes in the spot market, TSO decides on imports/export flows $\{f_j\}_{j\in N\cup I}$ as long as there are available transmission possibilities and it maximizes its profits from the transmission of electricity. Specifically, TSO effectively acts as an arbitrageur. TSO's problem is given by (3.6):

$$
\begin{aligned}
\max_{f} \quad & \sum_{j\in\{N\cup I\}} p_j f_j \\
s.t. \quad & \\
Cons\_TSO: \quad & \boxed{\begin{array}{lll} \sum_{j\in\{N\cup I\}} f_j = 0 & (\rho) & \\ \sum_{j\in\{N\cup I\}} PTDF_{l,j} f_j \le h_l & (\lambda_l^+) & \forall l \\ -\sum_{j\in\{N\cup I\}} PTDF_{l,j} f_j \le h_l & (\lambda_l^-) & \forall l, \end{array}}
\end{aligned}
\tag{3.6}
$$

where ρ, λ_l^+, and λ_l^- are Lagrange multipliers of problem (3.6). In (3.6), $Cons\_TSO$ is the set of Kirchoff law based transmission constraints faced by TSO in the electricity network. TSO is also a price-taker and cannot affect the nodal prices, p, to maximize its profit.

Finally, the nodal prices are determined to clear the spot market when supply matches the demand minus possible curtailments. In case of a curtailment, the electricity is priced at VOLL by the regulator. The spot market clearance conditions are given in (3.7):

$$
MCP\_Market: \boxed{\begin{array}{llll} 0 \le \sum_{g\in G}\sum_{k\in K_g} y_{jk}^g + \delta_j + f_j - d_j & \perp & p_j \ge 0 & \forall j\in\{N\cup I\} \\ 0 \le VOLL - p_n & \perp & \delta_n \ge 0 & \forall n\in N, \end{array}}
\tag{3.7}
$$

where p_j represents the locational marginal price of unit power (\$/MWh) at node $j\in\{N\cup I\}$ and δ_n is the curtailed energy. Note that node j may be both a supply and a demand node. If it is only a supply node, then δ_j and d_j equal to zero or if it is only a demand node then $(y^g_{jk})_{g\in G,k\in K}$ equal to zero.

The spot market equilibrium conditions consist of the KKT optimality conditions of problem (3.5) for all firms $g \in G$, the KKT optimality conditions of TSO's problem (3.6), and the market clearance conditions (3.7), which can altogether be formulated by the mixed complementarity problem (MCP) (3.8). An equilibrium point satisfying the conditions in MCP (3.8) consists of the optimal generation quantities y^* for all firms and the optimal import/export flow decisions f^* for TSO in the spot market:

$$
\begin{aligned}
&MCP\_Firms(x): && \boxed{\begin{array}{lll} 0 \le c^g_{ik} - p^*_i + \beta^{*g}_{ik} \quad \perp \quad y^{*g}_{ik} \ge 0 & \forall g \in G, i \in I_g, k \in K_g \\ 0 \le x^{*g}_{ik} - y^{*g}_{ik} \quad \perp \quad \beta^{*g}_{ik} \ge 0 & \forall g \in G, i \in I_g, k \in K_g \end{array}} \\
&MCP\_TSO: && \boxed{\begin{array}{lll} 0 \le h_l - \sum_{j\in\{N\cup I\}} PTDF_{l,j} f^*_j & \perp \quad \lambda^{*+}_l \ge 0 & \forall l \\ 0 \le h_l + \sum_{j\in\{N\cup I\}} PTDF_{l,j} f^*_j & \perp \quad \lambda^{*-}_l \ge 0 & \forall l \\ p^*_j - \rho^* + \sum_{l\in L} PTDF_{l,j}(\lambda^{*-}_l - \lambda^{*+}_l) = 0 & & \forall j \in \{N\cup I\} \\ \sum_{j\in\{N\cup I\}} f^*_j = 0 & & \end{array}} \\
&MCP\_Market: && \boxed{\begin{array}{lll} 0 \le \sum_{g\in G}\sum_{k\in K_g} y^{*g}_{jk} + \delta^*_j + f^*_j - d_j & \perp \quad p^*_j \ge 0 & \forall j \in \{N\cup I\} \\ 0 \le VOLL - p^*_n & \perp \quad \delta^*_n \ge 0 & \forall n \in N. \end{array}}
\end{aligned}
\tag{3.8}
$$

Boucher and Smeers (2001) consider a competitive equilibrium of a game in spot market where none of the agents (firms, consumers, and TSO) has market power. The interactions between firms and TSO in (3.5) and (3.6), respectively, is an example of such a game. Boucher and Smeers (2001) also introduce an optimization problem referred to as Optimal Power Flow Problem (OPF). In our setting, for given x, the OPF problem with exogenous demand corresponds to a linear program (LP) as given in (3.9):

$$
\begin{aligned}
&\min_{\{y,\delta,f\}} && \sum_{g\in G}\sum_{i\in I_g}\sum_{k\in K_g} c^g_{ik} y^g_{ik} + VOLL \sum_n \delta_n \\
&s.t. && \\
&Cons\_Market: && \boxed{\begin{array}{ll} \sum_{g\in G}\sum_{k\in K_g} y^g_{jk} + \delta_j + f_j \ge d_j \quad (p_j) & \forall j \in \{N\cup I\} \\ \delta_n \ge 0 & \forall n \in N \end{array}} \\
& && f \quad \text{satisfy } Cons\_TSO \\
& && y^g \quad \text{satisfy } Cons\_g(x^g) \qquad \forall g \in G.
\end{aligned}
\tag{3.9}
$$

Let (y^*, δ^*) and p^* be the optimal primal solution (generation quantities, curtailed demand) and optimal dual solution (nodal prices) of OPF problem (3.9) for a given x, respectively. Boucher and Smeers (2001) show that the solution (y^*, δ^*, p^*) is also a competitive equilibrium of the game between firms and TSO defined in (3.8) and vice versa. Indeed, it is easy to verify that the set of necessary and sufficient optimality conditions of the LP (3.9) is equivalent to the MCP (3.8). Therefore, we may solve the LP (3.9) directly and take its optimal solution as a perfect competition equilibrium of the spot market at the second stage.

First Stage: At the first stage, each firm $g \in G$ determines its optimal investment quantities x^g maximizing its long term profit which is equal to its optimal short term profit from the spot market at the second stage minus its investment cost. Since firms are price-takers, they act as if the market price is given, and hence the nodal prices, p, appear as parameters in firms' both first and second stage problems (3.10) and (3.5), respectively. Given p, each firm $g \in G$ maximizes its long term profit by optimization problem (3.10) at the first stage:

$$\max_{x^g \geq 0} \quad \sum_{i \in I_g} \sum_{k \in K_g} (p_i - c^g_{ik}) y^{*g}_{ik}(x^g) - \sum_{i \in I_g} \sum_{k \in K_g} \kappa_k x^g_{ik} \tag{3.10}$$

where $\{y^{*g}_{ik}(x^g),\ \forall g, i, k\}$ are the optimal generation quantities of firms in the spot market at equilibrium for given $(x^g)_{g \in G}$. Next, in Lemma 3.3.1, we show that each firm's optimal investment is equal to its optimal generation amount in the spot market at equilibrium when we have constant demand[5]. This is an intuitive result for deterministic investment problems with fixed demand level; however we are not aware of a formal proof.

Lemma 3.3.1. *Let x^{*g} be the vector of optimal investment quantities of each firm $g \in G$ for (3.10) and y^* be the vector of optimal generation quantities from OPF problem (3.9) for $x = x^*$. Then $y^{*g} = x^{*g}$ for all $g \in G$.*

Proof. There are two possibilities for y^{*g} as a solution of OPF problem for $x = x^*$: $y^{*g}_{ik} = x^{*g}_{ik}$, or $y^{*g}_{ik} < x^{*g}_{ik}$. The latter cannot hold at optimum of the first stage problem of firm $g \in G$, since one can always decrease x^{*g}_{ik} to the level of y^{*g}_{ik} and achieve a higher profit. In other words, the latter is always dominated by the former which achieves the same cost for OPF problem and a higher profit for firm $g \in G$. ∎

In the next lemma, we provide a characterization for the optimality conditions of each firm's capacity decisions at the first stage. This type of formulation for deterministic problems has also been formulated in the literature and explicitly illustrates the impact of the scarcity rents determined at the second stage on the firm's investment decisions at the first stage. We see that firms have an incentive to invest if the scarcity rents determined at the second stage offset their investment cost. This is a very intuitive result which can also be observed in early capacity expansion models developed during regulatory periods. A corresponding result (with the expectation of scarcity rents) will appear later when firms choose their capacity under demand uncertainty in Section 3.3.2.1, as well as when the firms have market power at the first stage in Chapter 4.

[5]Note that this result will not hold for all periods and all firms when we have multiple demand periods, see Section 3.3.2 for details.

Lemma 3.3.2. *Let $x^* = \{x^{*g}\}_{g\in G}$ be a point such that lower level problem (3.9) has a feasible solution and $\Pi_g^*(x^g)$ is finite for all $g \in G$ in the neighborhood of x^*. Then x^* is an equilibrium of the first-stage game if and only if there exists β^* such that*

$$0 \le -\beta_{ik}^{*g} + \kappa_k \quad \perp \quad x_{ik}^{*g} \ge 0 \quad \forall g \in G, i \in I_g, k \in K_g. \tag{3.11}$$

Proof. Firm g's problem (3.10) can also be formulated as follows:

$$\max_{x^g \ge 0} \quad \Pi_g^*(x^g) - \sum_{i\in I_g}\sum_{k\in K_g} \kappa_k x_{ik}^g$$

where $\Pi_g^*(x^g)$ is the optimal objective function value for problem (3.5) at a given x^g and given prices. Note that (3.5) is a linear program where x^g is the right hand side parameter. It is well known that $\Pi_g^*(\cdot)$ is a concave function of x^g. $\Pi_g^*(x^g)$ is also subdifferentiable at x^{*g} when $\Pi_g^*(x^g)$ is finite in the neighborhood of x^*. Hence, x^{*g} is an optimal solution of (3.10) for each firm $g \in G$ if and only if there exist $\beta_{ik}^{*g} \in \dfrac{\partial \Pi_g^*(x^{*g})}{\partial x_{ik}^g}$ satisfying the necessary and sufficient optimality conditions

$$0 \le -\beta_{ik}^{*g} + \kappa_k \quad \perp \quad x_{ik}^{*g} \ge 0 \quad \forall g \in G, i \in I_g, k \in K_g. \blacksquare$$

Therefore, a solution to this two-stage game, if it exists, should satisfy the optimality conditions given in (3.11) at the first stage and the optimality conditions given in (3.8) at the second stage simultaneously, and vice versa. Combining these with Lemma 3.3.1, we obtain the next lemma.

Lemma 3.3.3. *If there exists a solution x^* to the two-stage game then it satisfies the following complementarity conditions:*

$$\begin{array}{rcll}
0 \le c_{ik}^g - p_i^* + \kappa_k & \perp & x_{ik}^{*g} \ge 0 & \forall g \in G, i \in I_g, k \in K_g \\
0 \le \sum_{g\in G}\sum_{k\in K_g} x_{jk}^{*g} + \delta_j^* + f_j^* - d_j & \perp & p_j^* \ge 0 & \forall j \in \{N \cup I\} \\
0 \le VOLL - p_n^* & \perp & \delta_n^* \ge 0 & \forall n \in N \\
(f^*, p^*, \rho^*, \lambda^{*+}, \lambda^{*-}) & \textit{satisfy} & MCP_TSO. &
\end{array} \tag{3.12}$$

Moreover, if there exists a solution to the complementarity conditions in (3.12), then it is a solution to the two-stage game.

Proof. See Appendix 3.A.

3.3.1.2 An Equivalent Single Optimization Problem

It is easy to see that since we will always have $x^* = y^*$ in the two-stage game with constant demand, (3.10) and (3.5) reduce to the following single-stage formulation:

$$
\begin{aligned}
\bar{\Pi}_g^*(x^g) := \max_{x^g} \quad & \sum_{i \in I_g} \sum_{k \in K_g} (p_i - c_{ik}^g - \kappa_k) x_{ik}^g \\
s.t. \quad & x_{ik}^g \geq 0 \qquad\qquad \forall i \in I_g, k \in K_g.
\end{aligned}
\tag{3.13}
$$

The market clearance conditions can also be modified such that y^g in (3.7) is replaced with x^g (say in (3.7′)). Then, one can easily verify that the equilibrium conditions of the resulting single-stage game ((3.13), (3.6), and $(3.7)'$) are equivalent to the complementarity conditions given in (3.12); therefore by using Lemma 3.3.3, the solution of this single-stage game is equal to the solution of the two-stage game in Section 3.3.1.1. In other words, in perfect competition open loop and closed loop equilibria coincide.

Next, we show that a perfect competition equilibrium of the two-stage (or one-stage) game can be found by solving a particular single optimization problem which we introduce below in (3.14). This formulation is nothing but a variation of early capacity expansion models, given in (3.1), used for decisions of regulated monopolies. In this formulation, one can also think that the investment amounts of all firms are decided by a central decision maker or a regulated monopoly who is minimizing the total cost of the system (i.e., total generation, investment, and dispatch costs):

$$
\begin{aligned}
\min_{\{x,\delta,f\}} \quad & \sum_{g \in G} \sum_{i \in I_g} \sum_{k \in K_g} (c_{ik}^g + \kappa_k) x_{ik}^g + VOLL \sum_n \delta_n \\
s.t. \quad & \sum_{g \in G} \sum_{k \in K_g} x_{jk}^g + \delta_j + f_j \geq d_j \qquad (p_j) \quad \forall j \in \{N \cup I\} \\
& x, \delta \geq 0 \\
& f \text{ satisfy } Cons\_TSO.
\end{aligned}
\tag{3.14}
$$

Theorem 3.3.4. *A solution to the optimization problem (3.14), if it exists, is a solution of the two-stage game. Moreover, if there exists a solution of the two-stage game, then it is also a solution of the optimization problem (3.14).*

Proof. The necessary and sufficient optimality conditions of the linear program in (3.14) is equivalent to the complementarity conditions given in (3.12). By using Lemma 3.3.3, the result follows immediately. ∎

By using Theorem 3.3.4, the uniqueness and existence of competitive equilibria for the two-stage game in Section 3.3.1.1 may also be established. The existence follows from the existence of a solution to the optimization problem (3.14), i.e., if it is feasible and bounded. Moreover, if the solution to (3.14) is unique, then clearly there is a unique perfect competition equilibrium to the two-stage game.

3.3.2 Two-stage Equilibrium Model with Stochastic Exogenous Demand

In Section 3.3.1, we focused on a constant demand load over a year and showed that single (open loop) and two-stage (closed loop) games are equivalent; furthermore, one can find the perfect competition equilibrium by solving a single optimization problem. We have done this to present the notation and the basic properties of the underlying mathematical model. Clearly, demand is seasonal and time varying in reality. Moreover, future demand is uncertain. Both seasonality and uncertainty of future demand affect the choice of the plant type. While a constant yearly demand would lead to selecting a single technology in the solution of (3.14), seasonality and uncertainty of demand imply a portfolio of technologies. We thus extend the preceding model and consider the more realistic case in which demand is uncertain and varies, say, over a year. First, we prove that one can find a perfect competition equilibrium to the two-stage game under demand uncertainty by solving a two-stage stochastic program. We then show that the equivalence of single and two-stage games still holds in a perfectly competitive market when demand is random and has finite number of possible scenarios. In other words, the open loop equilibria and the closed loop equilibria coincide.

In Section 3.3.2.1, we analyze the solution of the two-stage competitive game outlined in Section 3.3.1.1 under demand uncertainty. We then introduce in Section 3.3.2.2 a two-stage stochastic program where a central decision maker decides on the capacity levels of all firms minimizing the total expected cost at the upper level under demand uncertainty. He/she then chooses the optimal generation quantities of all firms as demand is observed at the lower level. The spirit of this stochastic program is not very far away from the early capacity expansion model given in (3.1). We end Section 3.3.2.2 with Theorem 3.3.7 by showing that a solution of this two-stage stochastic program is also a solution of the two-stage stochastic game. Finally, in Section 3.3.2.3 we give the single-stage formulation of the two-stage stochastic game when the random demand has discrete distribution. We show that an equilibrium of the single-stage formulation is also an equilibrium of the two-stage formulation

3.3.2.1 The Perfect Competition Equilibrium under Demand Uncertainty

Consider now the case when the investment decision at the first stage should be made before observing the uncertain demand at the second stage. The notation of the two-stage model under uncertainty will be almost identical to the notation given in Section 3.2 except we will utilize $\omega \in \Omega$ to denote the uncertainty in demand that can take different values in different states of the world $\omega \in \Omega$, each occurring with some probability. We will also denote the dependency of the second stage variables with respect to ω in order to facilitate the main distinction between the first stage variables and the second stage variables where the former do not depend on $\omega \in \Omega$. To derive our theoretical results, we will assume that the probability distribution of demand, $d(\omega)$, is known. This assumption is valid in situations where the dispatch of electricity and market clearance in the spot market repeats itself and the distribution of demand can be estimated from historical data. In practice, also as part of our numerical procedures in Section 3.8, we will need only a sample of $d(\omega)$ rather than the entire distribution of $d(\omega)$. Without knowing the exact distribution of the random variable, samples of $d(\omega)$ may be obtained simply from historical data or, for instance, from computation-based simulations (where it may be easier to estimate the so-called basic factors, but since these factors interact in nonlinear and/or non-smooth ways, numerical procedures are needed to draw the samples).

Second Stage: For now, we suppose that $d_n(\omega)$ indicates continuous random demand at node n with a general distribution Ψ_n. Let $(\Omega, \mathscr{F}, \Psi)$ denote the common underlying probability space where Ψ represents the joint probability distribution for the random demand vector $d(\omega) := \{d_n(\omega)\}_{n \in N}$ with $E[|d(\omega)|] < \infty$. Then for given $\omega \in \Omega$, we can write the second stage problem of each firm $g \in G$ in (3.5), TSO's problem in (3.6), and the market clearing conditions in (3.7) in the state of the world ω (i.e., with second stage variables depending on ω). For instance for given x^g, the second stage problem of each firm $g \in G$ in state of the world ω is given as:

$$
\begin{aligned}
\Pi_g^*(\omega, x^g) := \max_{y^g(\omega)} \quad & \sum_{i \in I_g} \sum_{k \in K_g} (p_i(\omega) - c_{ik}^g) y_{ik}^g(\omega) \\
s.t. \quad & \\
Cons\_g(\omega, x^g): \quad & \boxed{\begin{array}{lll} y^g(\omega) \le x_{ik}^g & (\beta_{ik}^g(\omega)) & \forall i \in I_g, k \in K_g \\ y_{ik}^g(\omega) \ge 0 & & \forall i \in I_g, k \in K_g, \end{array}}
\end{aligned}
\tag{3.15}
$$

which is identical to (3.5) when there is one state of the world with constant demand. Note that since all firms are assumed to be price-takers, the nodal prices, $p(\omega)$, appear as parameters in firms' both first and second stage problems.

By similar arguments in Section 3.3.1.1, we know that we can find a perfect competition equilibrium of the spot market at each ω by solving OPF problem (3.16). For given x and each $\omega \in \Omega$:

$$
\begin{aligned}
Z^*(\omega, x) \quad & := \min_{\{y(\omega), \delta(\omega), f(\omega)\}} \sum_{g \in G} \sum_{i \in I_g} \sum_{k \in K_g} c_{ik}^g y_{ik}^g(\omega) + VOLL \sum_n \delta_n(\omega) \\
& s.t. \\
Cons\_Market(\omega) \quad & : \boxed{\begin{array}{lll} \sum_{g \in G} \sum_{k \in K_g} y_{jk}^g(\omega) + \delta_j(\omega) + f_j(\omega) \ge d_j(\omega) & (p_j(\omega)) & \forall j \in \{N \cup I\} \\ \delta_n(\omega) \ge 0 & & \forall n \in N \end{array}}
\end{aligned}
\tag{3.16}
$$

$$
Cons\_TSO(\omega) \quad : \quad \boxed{\begin{array}{lll} \sum_{j\in\{N\cup I\}} f_j(\omega) = 0 & (\rho(\omega)) & \\ h_l - \sum_{j\in\{N\cup I\}} PTDF_{l,j} f_j(\omega) \geq 0 & (\lambda_l^+(\omega)) & \forall l \\ h_l + \sum_{j\in\{N\cup I\}} PTDF_{l,j} f_j(\omega) \geq 0 & (\lambda_l^-(\omega)) & \forall l \end{array}}
$$

$$
Cons\_g(\omega, x^g) \quad : \quad \boxed{\begin{array}{lll} x_{ik}^g - y_{ik}^g(\omega) \geq 0 & (\beta_{ik}^g(\omega)) & \forall g \in G, i \in I_g, k \in K_g \\ y_{ik}^g(\omega) \geq 0 & & \forall i \in I_g, k \in K_g. \end{array}} \quad \forall g \in G
$$

Remark 3.3.5. Note that (3.16) is almost identical to (3.9) in which we indicate the explicit dependence to ω for the variables that are affected by demand uncertainty. From Boucher and Smeers (2001), we know that at each ω a solution of (3.16) is also a perfect competition equilibrium of the spot market at the second stage.

First Stage: At the first stage, we assume risk neutral firms making decisions on the basis of the expectation of their short term profit in the spot market. Consequently, we can write the optimization problem of each risk neutral firm $g \in G$ maximizing its long term profit at the first stage:

$$
\max_{x^g \geq 0} \quad E_\omega[\sum_{i\in I_g}\sum_{k\in K_g}(p_i(\omega) - c_{ik}^g) y_{ik}^{*g}(\omega, x^g)] - \sum_{i\in I_g}\sum_{k\in K_g} \kappa_k x_{ik}^g, \tag{3.17}
$$

where $y^*(\omega, x)$ is the vector of generation quantities of firms in the spot market at equilibrium and, hence is the solution of Optimal Power Flow Problem (3.16) at the second stage for given x and $\omega \in \Omega$.

It is obvious that we no longer have the equality of first and second stage decision variables as in Lemma 3.3.1 when we move to the two-stage game under uncertainty. However, one can still use arguments similar to the ones in *Proof of Lemma 3.3.2* in order to formulate the impact of average scarcity rent received at the second stage on the investment decisions of the firms.

Lemma 3.3.6. *Let $\Pi_g^*(\omega, x^g)$ be finite at the neighborhood of a point $x^* = \{x^{*g}\}_{g\in G}$ for almost every $\omega \in \Omega$. For investment choice x^{*g}, let $E_\omega[\beta_{ik}^{*g}(\omega)]$ be the expected scarcity rent that firm $g \in G$ receives at the second stage for using technology $k \in K_g$ at node $i \in I_g$. Then x^* is a solution of the first stage game if and only if*

$$
0 \leq -E_\omega[\beta_{ik}^{*g}(\omega)] + \kappa_k \quad \perp \quad x_{ik}^{*g} \geq 0 \quad \forall g \in G, i \in I_g, k \in K_g. \tag{3.18}
$$

Proof. See Appendix 3.A.

One can interpret the expected scarcity rent in (3.18) as the expected marginal revenue of firm g at node i for investing in technology k. If the expected marginal revenue of investing in technology k at node i is not enough to cover the firm's marginal cost of investment in technology k (κ_k), the firm chooses not to invest in that technology at node i. Otherwise, the firm invests in technology k at node i at a level where the expected scarcity rent is equal to the unit investment cost. (3.18) is essentially identical to the relation (3.4) of early capacity expansion models. For peak load generators, $\beta_{ik}^{*g}(\omega)$ is equal to zero unless demand is curtailed. Thus, peak load generators tend to underinvest so that total generation capacity is below the peak load with a certain probability and they cover their investment costs by VOLL during peak hours. Note that (3.18) represents the first stage equilibrium conditions of risk neutral generators. When generators are assumed to be risk averse, the formulation in (3.18) can be modified by replacing the statistical probabilities with the risk adjusted probabilities; see Ehrenmann and Smeers (2011) for the corresponding modification.

3.3.2.2 An Equivalent Two-stage Stochastic Program

In this section we show that one can find an equilibrium to the two-stage stochastic game by simply solving a stochastic program. As a consequence, the computational challenge of finding a solution of the two-stage game may be considerably reduced. The stochastic program we present below may be considered as the capacity expansion problem of a central decision maker who chooses the capacities (x) of all firms at the first stage in order to minimize the total expected cost of the system without knowing the future uncertain demand. He/she then determines the dispatch quantities $y(\omega)$ of all firms after observing the demand (possibly repeatedly) at the second stage by solving (3.16). The problem faced by the central decision maker at the first stage is formulated as

$$\min_{x\geq 0} \quad E_\omega[Z^*(\omega,x)] + \sum_{g\in G}\sum_{i\in I_g}\sum_{k\in K_g} \kappa_k x_{ik}^g. \tag{3.19}$$

Note that in Section 3.3.2.1 we have an equilibrium problem at the first stage consisting of $\mid G \mid$ optimization problems, each one given by (3.17) for $g \in G$. The stochastic program we introduce here consists of a single optimization problem, namely (3.19), at the first stage and the decision variables are the investment quantities of all firms. At the second stage, we have another single optimization problem for each realization which is formulated by (3.16).

Theorem 3.3.7. *Consider the two-stage stochastic program which consists of the problems* (3.19) *at the first stage and* (3.16) *at the second stage. Let x^* be the optimal solution of this two-stage stochastic program where $Z^*(\omega,x)$ and $\Pi_g^*(\omega,x^g)$ are finite in the neighborhood of x^* for almost every $\omega \in \Omega$. Then x^* is also a perfect competition equilibrium of the two-stage stochastic game given in Section 3.3.2.1 and vice versa.*

Proof. (3.16) is a linear program. Thus, we know that $Z^*(\omega,x)$ is a convex function of x for all $\omega \in \Omega$ which implies the convexity of the expectation $E_\omega[Z^*(\omega,\cdot)]$. By using an argument similar to the one in the *Proof of*

Lemma 3.3.6, one can show that $E_\omega[\beta_{ik}^{*g}(\omega)]$ is unique, $E_\omega[Z^*(\omega,\cdot)]$ is differentiable, and

$$\frac{\partial E_\omega[Z^*(\omega,\cdot)]}{\partial x_{ik}^g} = -E_\omega[\beta_{ik}^{*g}(\omega)]. \tag{3.20}$$

Since $E_\omega[Z^*(\omega,\cdot)]$ is convex and differentiable, x^* is optimal for the problem (3.19) if and only if

$$0 \le \frac{\partial E_\omega[Z^*(\omega,x^*)]}{\partial x_{ik}^g} + \kappa_k \quad \perp \quad x_{ik}^{*g} \ge 0 \quad \forall g \in G, i \in I_g, k \in K_g.$$

By using (3.20), we can rewrite the necessary and sufficient optimality conditions of (3.19) as

$$0 \le -E_\omega[\beta_{ik}^{*g}(\omega)] + \kappa_k \quad \perp \quad x_{ik}^{*g} \ge 0 \quad \forall g \in G, i \in I_g, k \in K_g. \tag{3.21}$$

The necessary and sufficient optimality conditions in (3.21) are identical to the equilibrium conditions of the first stage game given in Lemma 3.3.6. Besides, the necessary and sufficient optimality conditions of (3.16) are the equilibrium conditions of the second stage game. The solution $(x^*, y^*(\omega,x^*))$ to the two-stage stochastic program is feasible for the first stage game and an equilibrium of the second stage game since it satisfies the optimality conditions of (3.16). Moreover, it is also an equilibrium of the first stage game since it satisfies the equilibrium conditions at the first stage by equivalence of (3.21) and (3.18).

One can also make the same argument for the opposite direction. If $(x^*, y^*(\omega,x^*))$ is an equilibrium of the two-stage stochastic game, then it is a solution to the complementarity problem (3.18) and it satisfies the optimality conditions of (3.16). Since (3.18) is equivalent to (3.21) and the second stage optimality conditions of both the stochastic program and the stochastic game are derived from the same problem (3.16), $(x^*, y^*(\omega,x^*))$ will also be a solution of the two-stage stochastic program. ∎

As a conclusion, under the perfect competition assumption one can solve the stochastic optimization problem (3.19) of the central decision maker and take its optimal solution as equilibrium point of the two-stage stochastic game defined in Section 3.3.2.1.

3.3.2.3 Equivalence of Open and Closed Loop Equilibria with Finite Number of Scenarios

In this section, we assume that the demand distribution Ψ has a finite support (e.g., set of time segments) and takes values $d(\omega_1), d(\omega_2), \ldots, d(\omega_M)$ with respective probabilities $\pi_1, \pi_2, \ldots, \pi_M$ (e.g., duration of time segments). Obviously, one could also view $d(\omega_1), d(\omega_2), \ldots, d(\omega_M)$ as a particular sample of the random variable $d(\omega)$ with a general distribution. Then,

$$\begin{aligned} E_\omega[\Pi_g^*(\omega, x^g)] &= E_\omega[\sum_{i \in I_g} \sum_{k \in K_g} (p_i(\omega) - c_{ik}^g) y_{ik}^{*g}(\omega, x^g)] - \sum_{i \in I_g} \sum_{k \in K_g} \kappa_k x_{ik}^g \\ &= \sum_{m=1}^{M} \pi_m \sum_{i \in I_g} \sum_{k \in K_g} (p_i(\omega_m) - c_{ik}^g) y_{ik}^{*g}(\omega_m, x^g) - \sum_{i \in I_g} \sum_{k \in K_g} \kappa_k x_{ik}^g \end{aligned}$$

is the first stage objective function (3.17) of each firm $g \in G$ where $p(\omega_m)$ is given. Let $y^*(\omega_m, x)$ be the solution of the second stage OPF problem (3.16) for the scenario ω_m. The equilibrium conditions (3.18) of the first stage game can easily be restated as:

$$0 \leq -\sum_{m=1}^{M} \pi_m \beta_{ik}^{*g}(\omega_m) + \kappa_k \quad \perp \quad x_{ik}^{*g} \geq 0 \quad \forall g \in G, i \in I_g, k \in K_g. \tag{3.22}$$

The equilibrium conditions of the second stage game can also be easily modified as the KKT optimality conditions of OPF (3.16) driven for each scenario. Next, we show that one can find an equilibrium point of the corresponding two-stage stochastic game with finite number of demand scenarios by solving a single-stage stochastic game. We first outline this single-stage game between the firms and TSO.

Each firm $g \in G$ chooses its optimal investment amount and generation quantities simultaneously such that it maximizes its total expected profit in (3.23):

$$\begin{aligned} \max_{\{x^g, y^g(\omega_m)\}} \quad & \sum_{m=1}^{M} \pi_m \sum_{i \in I_g} \sum_{k \in K_g} (p_i(\omega_m) - c_{ik}^g) y_{ik}^g(\omega_m) - \sum_{i \in I_g} \sum_{k \in K_g} \kappa_k x_{ik}^g \\ s.t. \quad & y^g(\omega_m) \text{ satisfy } Cons\_g(\omega_m, x^g) \qquad \forall m \in M \\ & x^g \geq 0, \end{aligned} \tag{3.23}$$

where $p(\omega_m)$ is the price observed by each firm in scenario ω_m. $p(\omega_m)$ is exogenous to each firm's problem and TSO's problem whereas it is endogenous to the MCP formulated by the KKT optimality conditions of all firms and TSO together with the market clearance conditions.

TSO's problem (3.24) is almost identical to (3.6) except it is formulated with explicit depends on ω_m since its import/export decisions depend on the state of the world (i.e., observed demand $d(\omega_m)$). For each scenario ω_m, TSO solves

$$\begin{aligned} \max_{f(\omega_m)} \quad & \sum_{j \in \{N \cup I\}} p_j(\omega_m) f_j(\omega_m) \\ s.t. \quad & f(\omega_m) \text{ satisfy } Cons\_TSO(\omega_m). \end{aligned} \tag{3.24}$$

Similarly, the spot market clearance conditions, $MCP\_Market(\omega_m)$, at each state of the world is almost identical to (3.7) except it is formulated with explicit depends on ω_m.

Next, we show in Theorem 3.3.8 that if the random demand has discrete distribution then we can find an equilibrium of the corresponding two-stage stochastic game in Section 3.3.2.1 by solving this single-stage stochastic game.

Theorem 3.3.8. *Let the demand distribution Ψ has a finite support and takes values $d(\omega_1), d(\omega_2), \dots, d(\omega_M)$ with respective probabilities $\pi_1, \pi_2, \dots, \pi_M$. Then an equilibrium of the single-stage stochastic game formulated by* (3.23), (3.24), *and MCP_Market(ω_m), if it exits, is also an equilibrium of the corresponding two-stage stochastic game given in Section 3.3.2.1 (by* (3.17) *and* (3.16)*). Moreover if there exists an equilibrium of this two-stage stochastic game, then it is also an equilibrium of the single-stage stochastic game.*

Proof. See Appendix 3.A.

To summarize, the energy-only market model presented in Section 3.3 provides positive margin for peak load generators when demand is curtailed and electricity price is set at VOLL. As a result, energy-only markets tend to lead to total generation capacity below the peak load with a certain probability. In reality VOLL is difficult to estimate in a direct way. An alternative is to assess the impact of a particular VOLL value on the probability of not meeting the load and to revise this value if this probability is not satisfactory (too high or too low, see Stoft (2002)). In case price caps lower than VOLL are used in practice, this may lead to higher frequency of curtailments and may enhance the resource adequacy problem. Hence, it may be necessary to resort to additional market mechanisms as remedies to create better incentives for capacity investment and operation. Next, we consider two different market designs, namely a capacity market in Section 3.4 and operating-reserve pricing in Section 3.5, as potential remedies to resource adequacy problem.

3.4 Imposing a Capacity Market

An alternative way to avoid resource adequacy problem is to implement a capacity market where the regulator sets a total capacity target based on historic data and expected demand and rewards a side payment to the firms who contribute to reach this target. We continue our analysis by including such a capacity market in our basic model of Section 3.3. In this modified model, the regulator imposes a capacity constraint on the total capacity which needs to be fulfilled at the time of investment. To account for this, the following modifications are needed:

(i) We impose the following market condition of the regulator at the first stage:

$$0 \leq \sum_{g,i,k} x_{ik}^g - H \quad \perp \quad \lambda \geq 0, \tag{3.25}$$

where H is the capacity target which may be estimated by the regulator on the basis of the forecasts of the demand fluctuations and λ may be considered as the price of the capacity which is paid to the firms who contribute to the sufficient investment level.

(ii) We also add the corresponding side payments for capacity to each firm's profit at the first stage:

$$\lambda \sum_{i,k} x_{ik}^{g}.$$

Mathematically, these modifications constitute a straight forward extension of the two-stage game in Section 3.3. When we have constant demand, the equilibrium (KKT optimality) conditions of the first stage game given in Lemma 3.3.2 can easily be modified to:

$$\begin{aligned} 0 \le -\beta_{ik}^{*g} - \lambda^* + \kappa_k \quad &\perp \quad x_{ik}^{*g} \ge 0 \quad \forall g \in G, i \in I_g, k \in K_g, \\ 0 \le \sum_{g,i,k} x_{ik}^{*g} - H \quad &\perp \quad \lambda^* \ge 0. \end{aligned} \tag{3.26}$$

In case of implementation of a capacity market, the result in Lemma 3.3.1 (which is established for energy-only markets) holds if $H \le \sum_j d_j$ whereas a similar type of result given in Theorem 3.3.4 always holds; that is, it is still possible to show that one can find an equilibrium of this two-stage game by solving a single optimization problem. This optimization problem is a slightly modified version of (3.14) and is given in (3.27):

$$\begin{aligned} \min_{\{x,y,\delta,f\}} \quad & \sum_{g \in G} \sum_{i \in I_g} \sum_{k \in K_g} (c_{ik}^{g} y_{ik}^{g} + \kappa_k x_{ik}^{g}) + VOLL \sum_{n} \delta_n \\ s.t. \quad & \sum_{g,i,k} x_{ik}^{g} - H \ge 0 \qquad\qquad (\lambda) \\ & x \quad \ge 0 \\ & (y,\delta,f) \quad \text{satisfy } Cons\_Market \\ & f \quad \text{satisfy } Cons\_TSO \\ & y^g \quad \text{satisfy } Cons\_g(x^g) \qquad \forall g \in G. \end{aligned} \tag{3.27}$$

In case of uncertain demand, a similar modification of the equilibrium conditions (3.18) in Lemma 3.3.6 can be done for the first stage game:

$$\begin{aligned} 0 \le -E_\omega[\beta_{ik}^{*g}(\omega)] - \lambda^* + \kappa_k \quad &\perp \quad x_{ik}^{*g} \ge 0 \quad \forall g \in G, i \in I_g, k \in K_g, \\ 0 \le \sum_{g,i,k} x_{ik}^{*g} - H \quad &\perp \quad \lambda^* \ge 0. \end{aligned} \tag{3.28}$$

Note that the equilibrium conditions of the second stage game in the future spot market do not change and are thus still formulated by the optimality conditions of the OPF problem (3.16). Therefore, by similar arguments used for the energy-only market, we can again prove that an equilibrium of this two-stage stochastic game with capacity market can be found by solving a two-stage stochastic program. This stochastic program given in (3.29) is a slightly modified version of (3.19) in Section 3.3.2.2 with the additional deterministic constraint $\sum_{g,i,k} x_{ik}^g \geq H$:

$$\begin{aligned} \min_{x\geq 0} \quad & E_\omega[Z^*(\omega,x)] + \sum_{g\in G}\sum_{i\in I_g}\sum_{k\in K_g} \kappa_k x_{ik}^g \\ s.t. \quad & \sum_{g,i,k} x_{ik}^g - H \geq 0 \qquad\qquad (\lambda). \end{aligned} \tag{3.29}$$

Hence, the two-stage stochastic program consists of an optimization problem, namely (3.29), which minimizes the total expected cost of the system subject to a single constraint at the first stage and the OPF problem (3.16) at the second stage.

3.5 Operating-reserve Pricing

A more sophisticated approach for the regulator to provide extra capacity payments to the firms contributing to a sufficient level of investment is operating-reserve pricing. The principle of operating-reserve pricing is providing the firms with extra regulated payment whenever the system's total operating-reserve is scarce.

Let $rp(r,x,d)$ be the value of operating-reserves determined by the regulator for a given value of total operating-reserves $r = ex - ey$ where e is the transpose of the vector of 1's of appropriate dimension. Once x, y, and d are observed in the spot market, a mark-up price of $rp(r,x,d)$ is computed by the regulator and then taken as exogenous reserve price by the firms. For given x and d, we assume the following properties of $rp(r,x,d)$.

Assumption 3.5.1. *$rp(r,x,d)$ is a monotone decreasing and differentiable function of total operating-reserves r where*

- *If $ed = ey \ll ex$, there is ample operating-reserve and there is no markup; that is $rp(r,x,d) = 0$.*

- *If $ed = ey \leq ex$ and ey is close to ex, then the operating-reserve is scarce and the regulator charges consumers with the extra price of $rp(r,x,d)$ in addition to the equilibrium price p.*

- *If $ex = ey < ed$, there is curtailment and the regulator sets the price to $VOLL$ (or to a price cap).*

Next, we modify the formulation of the two-stage game in Section 3.3 by incorporating operating-reserve pricing scheme. In Sections 3.5.1 and 3.5.2, we give the corresponding two-stage model formulations in deterministic and stochastic settings respectively and derive the equilibrium conditions at the first stage. Note that choosing the function $rp(r,x,d)$ is an important issue since it may change the structure of the underlying mathematical formulation of the model and consequently applicable solution methods. In Section 3.5.3, we elaborate on that issue and give two possible formulations of $rp(r,x,d)$ function. We show that depending on the formulation, one cannot always preserve the simplicity of the single optimization formulation in deterministic setting and the two-stage stochastic program in stochastic setting.

3.5.1 The Perfect Competition Equilibrium under Constant Demand

In this section, we modify both the first and the second stage problems given in Section 3.3.1 by incorporating the operating-reserve pricing.

Second Stage: Let γ denote the unit price of operating-reserves. It is exogenous to the firms since they are price-takers and is set by the regulator at equilibrium in the spot market. For a given γ, firm $g \in G$ receives additional revenue, $\sum_{i \in I_g} \sum_{k \in K_g} \gamma(x_{ik}^g - y_{ik}^g)$, for its operating-reserves at the second stage. This extra regulated payment received by firm $g \in G$ is added to its objective function at the second stage. The modified second stage problem of each firm $g \in G$ maximizing its short run profit is given as:

$$
\begin{aligned}
\Pi_g^{*R}(x^g) := \max_{y^g} \quad & \sum_{i \in I_g} \sum_{k \in K_g} (p_i - c_{ik}^g - \gamma) y_{ik}^g + \gamma \sum_{i \in I_g} \sum_{k \in K_g} x_{ik}^g \\
s.t. \quad & \\
Cons\_g(x^g): \quad & \boxed{\begin{array}{lll} y_{ik}^g \leq x_{ik}^g & (\beta_{ik}^g) & \forall i \in I_g, k \in K_g \\ y_{ik}^g \geq 0 & & \forall i \in I_g, k \in K_g, \end{array}}
\end{aligned}
\tag{3.30}
$$

where p and γ are exogenous parameters to each firm's problem whereas they are endogenous to the whole system and are set at a level where the spot market clears itself. To emphasize, $\gamma^* = rp(ex - ey^*, x, d)$ is the unit operating-reserve price set by the regulator for given x, d and optimal generation dispatch y^* in the spot market at equilibrium. In addition, TSO's problem and the market clearing conditions remain identical to (3.6) and (3.7), respectively.

One can write the necessary and sufficient KKT optimality conditions of (3.30) for each firm $g \in G$, the optimality conditions of (3.6) for TSO, and the market clearing conditions (3.7) and set $\gamma^* = rp(ex - ey^*, x, d)$. The resulting KKT conditions are equivalent to the MCP (3.31). Hence, the solution to MCP (3.31) is a competitive equilibrium of the spot market. The first complementarity constraint in (3.31) ensures that when a firm produces positive amount ($y_{ik}^{*g} > 0$), a price is paid, which covers its marginal cost plus the scarcity rent plus the operating-reserve price $rp(ex - ey^*, x, d)$:

$$
\begin{array}{llll}
0 \leq c_{ik}^g - p_i^* + \beta_{ik}^{*g} + rp(ex - ey^*, x, d) & \perp & y_{ik}^{*g} \geq 0 & \forall g \in G, i \in I_g, k \in K_g \\
0 \leq x_{ik}^g - y_{ik}^{*g} & \perp & \beta_{ik}^{*g} \geq 0 & \forall g \in G, i \in I_g, k \in K_g \\
(y^*, \delta^*, f^*, p^*, \rho^*, \lambda^{*+}, \lambda^{*-}) & \text{satisfy} & MCP\_Market \cap MCP\_TSO. &
\end{array}
\tag{3.31}
$$

As discussed in Section 3.3.1, we can formulate the corresponding OPF problem (3.32) whose solution gives perfect competition equilibrium in the spot market for given x:

$$
\begin{aligned}
\min_{\{y,\delta,f\}} \quad & \sum_{g\in G}\sum_{i\in I_g}\sum_{k\in K_g} c^g_{ik} y^g_{ik} + VOLL\sum_{n\in N}\delta_n - R(ex-ey,x,d) \\
s.t. \quad (y,\delta,f) \quad & \text{satisfy } Cons\_Market \\
f \quad & \text{satisfy } Cons\_TSO \\
y^g \quad & \text{satisfy } Cons\_g(x^g) \qquad \forall g\in G,
\end{aligned}
\tag{3.32}
$$

in which $R(ex-ey,x,d) := \int_0^{ex-ey} rp(s,x,d)ds$ can be interpreted as the additional welfare due to improved reliability, or willingness to pay for extra reliability; it is similar to the integral of the inverse demand function (e.g., in (3.2) or (3.42)) which is interpreted as consumers' willingness to pay. By Assumption 3.5.1, for given x and d, $rp(ex-ey,x,d)$ is a monotone decreasing function of $ex-ey$; therefore $R(ex-ey,x,d)$ is a concave function of $(ex-ey)$ and consequently it is concave in y. Hence OPF problem (3.32) is convex. The solution of OPF problem (3.32) is a competitive equilibrium of the modified game in the spot market since its necessary and sufficient KKT optimality conditions are equivalent to the MCP (3.31).

First Stage: Similar to Section 3.3.1, each firm $g\in G$ maximizes its long term profit at the first stage:

$$
\max_{x^g\geq 0} \quad \Pi^{*R}_g(x^g) - \sum_{i\in I_g}\sum_{k\in K_g}\kappa_k x^g_{ik}. \tag{3.33}
$$

When we consider the interaction between first and second stage problems, Lemma 3.3.1 does not hold anymore. Instead, we next show that optimality conditions of each firm's problem at the first stage now involves the scarcity rent and the operating-reserve price it receives at the second stage.

Theorem 3.5.2. *Let $x^* = \{x^{*g}\}_{g\in G}$ be such that OPF problem* (3.32) *has a feasible solution and $\Pi^{*R}_g(x^g)$ is finite for all $g\in G$ in the neighborhood of x^*. Then x^* is an equilibrium of the first-stage game if and only if there exists β^* and y^* such that*

$$
0 \leq -\beta^{*g}_{ik} - rp(ex^*-ey^*,x^*,d) + \kappa_k \quad \perp \quad x^{*g}_{ik} \geq 0 \quad \forall g\in G, i\in I_g, k\in K_g. \tag{3.34}
$$

Proof. See Appendix 3.A.

3.5.2 The Perfect Competition Equilibrium under Demand Uncertainty

We next formulate the two-stage game including operating-reserve pricing scheme when demand, $d_n(\omega)$, at each node n is random having a general distribution Ψ_n. Let $(\Omega, \mathscr{F}, \Psi)$ denote the common underlying probability space where Ψ represents the joint probability distribution for the random demand vector $d(\omega) := \{d_n(\omega)\}_{n\in N}$ with $E[|d(\omega)|] < \infty$.

Second Stage: Let $\gamma(\omega)$ be the unit price of reserved capacity at demand realization $\omega \in \Omega$. As mentioned earlier, it is exogenous to the firms at both stages since they are price-takers and is set by the regulator at a level where the spot market clears itself: $\gamma^*(\omega) = rp(ex - ey^*(\omega), x, d(\omega))$.

For given $\omega \in \Omega$, the second stage game is identical to its deterministic formulation given in Section 3.5.1 except with explicit dependence on ω. Hence, the arguments in Section 3.5.1 hold when demand is stochastic as well. That is, for each ω, $\Pi_g^{*R}(\omega, x^g)$ is a concave function of x^g and $\beta_{ik}^{*g}(\omega) + \gamma^*(\omega)$ is a subgradient of $\Pi_g^{*R}(\omega, x^g)$ at a given x^g. Moreover, one can solve the OPF problem (3.35) to find an equilibrium of the spot market at the second stage.

For given x and $\omega \in \Omega$, we have $R(ex - ey(\omega), x, d(\omega)) = \int_0^{ex-ey(\omega)} rp(s, x, d(\omega))ds$. Then the corresponding OPF problem at the second stage is formulated as:

$$\begin{aligned}
\min_{\{y(\omega),\delta(\omega),f(\omega)\}} \quad & \sum_{g\in G}\sum_{i\in I_g}\sum_{k\in K_g} c_{ik}^g y_{ik}^g(\omega) + VOLL \sum_{n\in N} \delta_n(\omega) - R(ex - ey(\omega), x, d(\omega)) \\
s.t. \quad (y(\omega), \delta(\omega), f(\omega)) \quad & \text{satisfy } Cons_Market(\omega) \\
f(\omega) \quad & \text{satisfy } Cons_TSO(\omega) \\
y^g(\omega) \quad & \text{satisfy } Cons_g(\omega, x^g) \qquad \forall g \in G.
\end{aligned} \tag{3.35}$$

First Stage: Assuming risk neutral firms, each firm $g \in G$ maximizes its long term expected profit at the first stage:

$$\max_{x^g \geq 0} \quad E_\omega[\Pi_g^{*R}(\omega, x^g)] - \sum_{i\in I_g}\sum_{k\in K_g} \kappa_k x_{ik}^g, \tag{3.36}$$

where, for given x^g, $\Pi_g^{*R}(\omega, x^g)$ is the optimal value of firm g's problem in the spot market(short term profit of firm $g \in G$) at realization $\omega \in \Omega$.

Similar to Lemma 3.3.6, it is possible to write the equilibrium conditions of the first stage game in terms of expected scarcity rent and operating-reserve prices that the firms receive at the second stage which is stated in the next theorem.

Theorem 3.5.3. *Let $\Pi_g^{*R}(\omega, x^g)$ be finite at the neighborhood of a point $x^* = \{x^{*g}\}_{g\in G}$ for almost every $\omega \in \Omega$. For investment choice x^{*g}, let $E_\omega[\beta_{ik}^{*g}(\omega) + rp(ex^* - ey^*(\omega), x^*, d(\omega))]$ be the expected marginal revenue that firm $g \in G$ receives at the second stage for using technology $k \in K_g$ at node $i \in I_g$. Then x^* is an equilibrium of the first stage game if and only if there exists $\beta^*(\omega)$ and $y^*(\omega)$ such that*

$$-E_\omega[\beta_{ik}^{*g}(\omega) + rp(ex^* - ey^*(\omega), x^*, d(\omega))] + \kappa_k \quad \perp \quad x_{ik}^{*g} \geq 0 \quad \forall g \in G, i \in I_g, k \in K_g. \tag{3.37}$$

Proof. See Appendix 3.A.

3.5.3 Operating-reserve Price Curve

Operating-reserve prices are based on predetermined reserve target levels set by the regulator. How to determine these target levels is an important issue since they would have an effect on the investment incentives. Moreover, the formulation of the $rp(\cdot)$ function would also effect the mathematical properties of the resulting two-stage game. Next, we give two different formulations for the $rp(\cdot)$ function. In the first one, the reserve targets are set by predetermined ratios of observed demand. In the latter one, the reserve targets are set by the predetermined ratios of installed capacity. We show that while the two-stage game with the first formulation is still convertible to a single-stage optimization problem in the deterministic setting and to a two-stage stochastic program in the stochastic setting, this simplicity is lost when the target levels depend on the installed capacity.

3.5.3.1 Setting Reserve Targets Based on Observed Demand

In this case, the regulator determines the reserve targets based on observed demand. Therefore, we take the operating-reserve price function as

$$rp(ex - ey, x, d) := rp\_demand(\frac{ex - ey}{ed}).$$

An example of the corresponding price curve for a total fixed demand level (ed), taken from Hogan (2005), is depicted in Figure 3.1. The x-axis denotes the percentage of over-capacity (or operating-reserves) with respect to the observed demand. For given x and d, $rp\_demand(\frac{ex-ey}{ed})$ is a piecewise linear decreasing function of operating-reserves. In the operating-reserve price curve of Hogan (2005), the critical operating-reserve levels predetermined by the regulator are the minimum level of reserve (3% of demand) and the nominal reserve target (7% of demand). The minimum level of reserves is set by the regulator to prevent a catastrophic failure through a widespread and uncontrolled blackout in the system. The regulator would not go below this level of reserves even if this required curtailment of inflexible demand. Above this minimum level, there would be more flexibility up to nominal reserve target (7%). This is the price-sensitive part of the operating-reserve price curve illustrated in Figure 3.1. In the range between 3% – 7%, as reserve levels approach the nominal target, the operating-reserve price would be decreasing.

Note that Hogan's operating-reserve price curve given in Figure 3.1 is piecewise linear. In our analysis, in order to preserve twice differentiability in OPF problems (3.32) and (3.35), we assume a differentiable operating-reserve price curve which is a smooth approximation of Hogan's curve. The details of this approximation using a sigmoid function are given in Section 3.9 and an example of the corresponding differentiable operating-reserve price curve is illustrated in Figure 3.1.

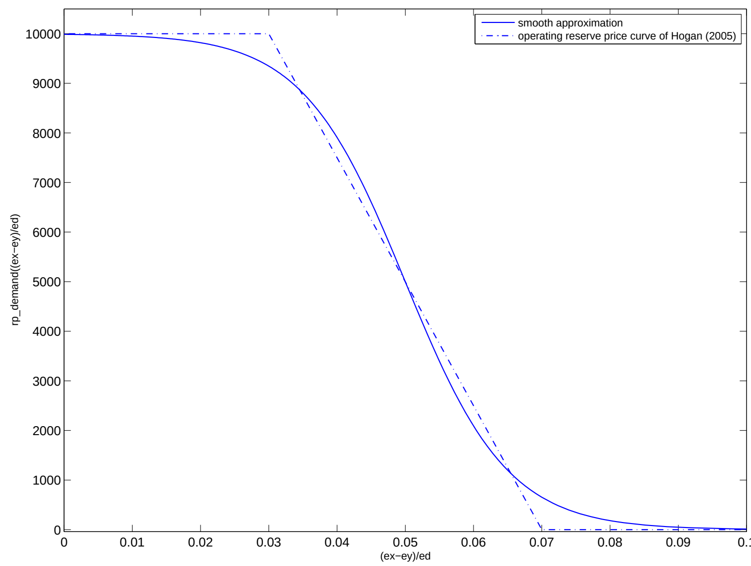


Figure 3.1: Operating-reserve price curve which is a smooth approximation of Hogan's curve on basis of observed demand

Next, for given x and d, we can calculate $R(ex-ey,x,d)$ as

$$\begin{aligned} R(ex-ey,x,d) \quad &:= R\_demand(\tfrac{ex-ey}{ed}) \\ &= \int_0^{ex-ey} rp\_demand(\frac{s}{ed})ds = ed\int_0^{\frac{ex-ey}{ed}} rp\_demand(s)ds. \end{aligned}$$

We know from Assumption 3.5.1 that $rp\_demand(\frac{ex-ey}{ed})$ is a monotone decreasing function of $\frac{ex-ey}{ed}$; therefore $R\_demand(\frac{ex-ey}{ed})$ is concave in $\frac{ex-ey}{ed}$ which is an affine function of (x,y). As we mentioned above, we assume that $rp\_demand(\frac{ex-ey}{ed})$ is differentiable; hence $R\_demand(\frac{ex-ey}{ed})$ is twice differentiable. Moreover, it is concave in (x,y) with

$$\frac{\partial R\_demand(\frac{ex-ey}{ed})}{\partial x^g_{ik}} = rp\_demand(\frac{ex-ey}{ed}), \text{ and} \tag{3.38}$$

$$\frac{\partial^2 R\_demand(\frac{ex-ey}{ed})}{\partial (x^g_{ik})^2} = \frac{\partial rp\_demand(\frac{ex-ey}{ed})}{\partial(\frac{ex-ey}{ed})}\frac{1}{ed} \le 0.$$

Remark 3.5.4. Joint concavity in (x,y) of $R\_demand(\frac{ex-ey}{ed})$ can be seen by constructing its Hessian matrix of and showing that it is negative semi-definite. The details of the proof are given in Appendix 3.A.

(3.38) indicates that a unit increase of x_{ik}^{g} indeed entails a marginal revenue for firm g from operating-reserves, which is equal to the operating-reserve price. By utilizing (3.38), we next show that a solution to (3.39) is a perfect competition equilibrium of the two-stage deterministic game:

$$
\begin{aligned}
\min_{\{x,y,\delta,f\}} \quad & \sum_{g\in G}\sum_{i\in I_g}\sum_{k\in K_g}(c_{ik}^{g}y_{ik}^{g}+\kappa_k x_{ik}^{g})+VOLL\sum_{n\in N}\delta_n - R\_demand(\frac{ex-ey}{ed}) \\
s.t. \quad & (y,\delta,f) \text{ satisfy } Cons\_Market \\
& f \text{ satisfy } Cons\_TSO \\
& y^g \text{ satisfy } Cons\_g(x^g) \qquad \forall g\in G \\
& x \geq 0.
\end{aligned}
\tag{3.39}
$$

Theorem 3.5.5. *A solution to the optimization problem* (3.39)*, if it exists, is a solution of the two-stage deterministic game with operating-reserve pricing. Moreover, if there exists a solution of the two-stage deterministic game, then it is also a solution of the optimization problem* (3.39).

Proof. The nonlinear program in (3.39) is convex in (x,y,δ,f). The necessary and sufficient optimality conditions of NLP (3.39) is equivalent to the first and second stage equilibrium conditions given in (3.34) and (3.31) respectively. The result follows immediately. ∎

Next, we obtain the corresponding result in the stochastic setting; that is, a solution to the two-stage stochastic program given in (3.40) is a perfect competition equilibrium of the two-stage stochastic game.

$$
\min_{x\geq 0} \quad E_{\omega}[Z^{*R}(\omega,x)]+\sum_{i\in I_g}\sum_{k\in K_g}\kappa_k x_{ik}^{g}, \tag{3.40}
$$

where

$$
\begin{aligned}
Z^{*R}(\omega,x) = \min_{\{y(\omega),\delta(\omega),f(\omega)\}} \quad & \sum_{g\in G}\sum_{i\in I_g}\sum_{k\in K_g} c_{ik}^{g}y_{ik}^{g}(\omega)+VOLL\sum_{n\in N}\delta_n(\omega) - R\_demand(\frac{ex-ey(\omega)}{ed(\omega)}) \\
s.t. \quad & (y(\omega),\delta(\omega),f(\omega)) \text{ satisfy } Cons\_Market(\omega) \\
& f(\omega) \text{ satisfy } Cons\_TSO(\omega) \\
& y^g(\omega) \text{ satisfy } Cons\_g(\omega,x^g) \qquad \forall g\in G.
\end{aligned}
$$

Theorem 3.5.6. *Let x^* be an optimal solution of the two-stage stochastic program formulated in* (3.40) *where $Z^{*R}(\omega,x)$ and $\Pi^{*R}(\omega,x^g)$ are finite in the neighborhood of x^* for almost every $\omega\in\Omega$. Then x^* is also a perfect competition equilibrium of the two-stage stochastic game with operating-reserve pricing and vice versa.*

Proof. By using the concavity of $R\_demand(\frac{ex-ey(\omega)}{ed(\omega)})$ in x, we know that $Z^{*R}(\omega,x)$ is a convex function of x for all $\omega \in \Omega$ which implies the convexity of $E_\omega[Z^{*R}(\omega,x)]$. One can compute the components of the gradient of $Z^{*R}(\omega,x)$ as:

$$\begin{aligned}\frac{\partial Z^{*R}(\omega,x)}{\partial x^g_{ik}} &= -\beta^{*g}_{ik}(\omega) - \frac{\partial R\_demand(\frac{ex-ey^*(\omega)}{ed(\omega)})}{\partial(\frac{ex-ey^*(\omega)}{ed(\omega)})} \cdot \frac{1}{ed} \\ &= -\beta^{*g}_{ik}(\omega) - rp\_demand(\tfrac{ex-ey^*(\omega)}{ed(\omega)}),\end{aligned}$$

except on a set L of Lebesgue measure zero.

By utilizing a similar argument in the *Proof of Theorem 3.3.7*, one can show that $E_\omega[Z^{*R}(\omega,x)]$ is differentiable and

$$\frac{\partial E_\omega[Z^{*R}(\omega,x^*)]}{\partial x^g_{ik}} = -E[\beta^{*g}_{ik}(\omega) + rp\_demand(\frac{ex^* - ey^*(\omega)}{ed(\omega)})].$$

Then one can easily derive the necessary and sufficient optimality conditions of the two-stage stochastic program (3.40) which are equivalent to the equilibrium conditions of the first stage game given in (3.37) and the optimality conditions of OPF problem (3.35). ∎

3.5.3.2 Setting Reserve Targets Based on Installed Capacity

Next, we deal with the case that the regulator determines the reserve targets based on total installed capacity. An example of the corresponding operating-reserve price curve for a fixed total installed capacity (ex) is given in Figure 3.2. This time the x-axis denotes the percentage of unused capacity with respect to the total available capacity. We assume the same minimum level of reserve (3% of total capacity) and the nominal reserve target (7% of total capacity) as in Figure 3.1. This time, working with an operating-reserve price curve like Figure 3.2 would mean that we take $rp(ey-ex,x,d) := rp\_capacity(\frac{ex-ey}{ex})$ which is again a differentiable function. Then, for given x, we can compute the corresponding willingness to pay for improved reliability as:

$$\begin{aligned}R(ex-ey,x,d) &:= R\_capacity(\tfrac{ex-ey}{ex}) \\ &= \int_0^{ex-ey} rp\_capacity(\frac{s}{ex})ds = ex\int_0^{\frac{ex-ey}{ex}} rp\_capacity(s)ds.\end{aligned}$$

Since $rp\_capacity(\frac{ex-ey}{ex})$ is differentiable, $R\_capacity(\frac{ex-ey}{ex})$ is twice differentiable. Moreover, $R\_capacity(\frac{ex-ey}{ex})$ is concave in (x,y) with

$$\frac{\partial R\_capacity(\frac{ex-ey}{ex})}{\partial x_{ik}^{g}} = \int_{0}^{\frac{ex-ey}{ex}} rp\_capacity(s)ds + rp\_capacity(\frac{ex-ey}{ex})\frac{ey}{ex}, \text{ and}$$

$$\frac{\partial^2 R\_capacity(\frac{ex-ey}{ex})}{\partial (x_{ik}^{g})^2} = \frac{\partial rp\_capacity(\frac{ex-ey}{ex})}{\partial(\frac{ex-ey}{ex})}\frac{ey^2}{ex^3} \leq 0, \ \forall x \geq 0, \tag{3.41}$$

where ey^2 is the sum of squares for all the elements of a vector y and ex^3 is the sum of cubes for all the elements of a vector x.

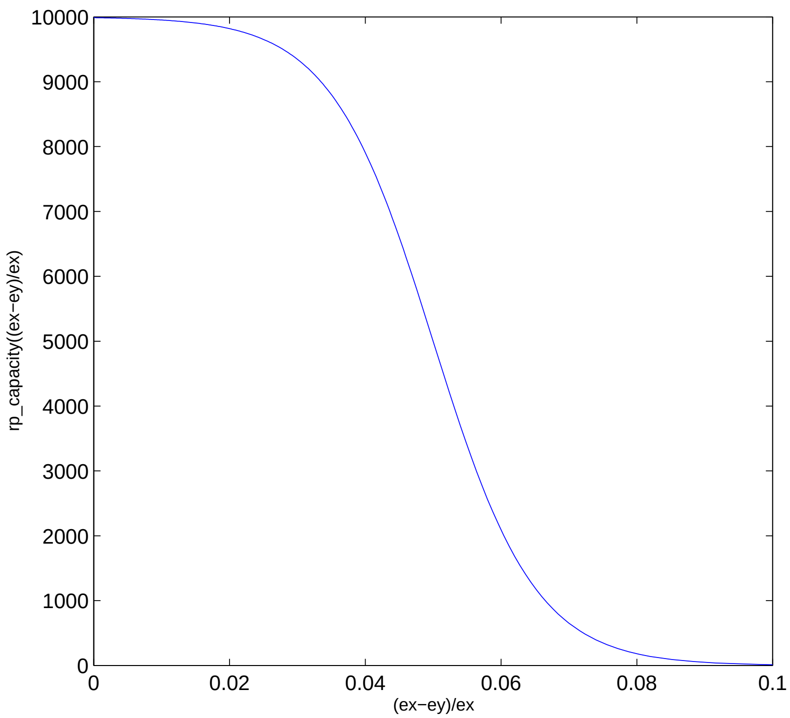


Figure 3.2: Operating-reserve price curve on basis of installed capacity

Remark 3.5.7. Similar to Remark 3.5.4, joint concavity in (x,y) of $R\_capacity(\frac{ex-ey}{ex})$ can be seen by constructing the Hessian matrix of $R\_capacity(\frac{ex-ey}{ex})$ and showing that it is negative semi-definite. The details of the proof are given in Appendix 3.A.

The first equation in (3.41) indicates that a unit increase of x_{ik}^{g} entails a marginal revenue from operating-reserves, which is different from the operating-reserve price. In other words, when we calculate the derivative of $R\_capacity$ with respect to x_{ik}^{g}, we do not get $rp\_capacity(\frac{ex-ey}{ex})$. Therefore, we cannot get the KKT conditions given in (3.11) from an optimization problem; hence, we can no longer reduce the two-stage game formulation to a single optimization problem in the deterministic setting and to a two-stage stochastic program in the stochastic setting. However, we can still formulate a mixed complementarity problem (MCP) which involves the equilibrium conditions of both first and second stage games and solve the corresponding MCP. We explain the details of solution methods for these problems in Section 3.8.

3.6 Demand-side Bidding (or an Endogenous Demand Curve)

In previous sections we consider three different ways of regulator's intervention to provide incentives for sufficient investment levels in electricity markets. Without the intervention of the regulator, the power systems tend to underinvest in generation capacity because of imperfections in the market due to unresponsive demand. When demand responds to price, regulator's intervention is needed less and the electricity market is more likely to operate normally such that the prices clear the market where supply meets the demand without any curtailments (e.g., Allcott (2012)). Therefore, if we consider "demand-side bidding" as a mechanism to ensure a price responsive demand, then it can be thought of as an alternative way of addressing the resource adequacy issue. In this section, we briefly comment on an electricity market where consumers can respond to prices.

When consumers respond to prices, we have an elastic demand and we represent the reaction of consumers to the prices by decreasing price-demand curves $p_n(d)$ with $p_n(0) < \infty$ for each demand node $n \in N$. On such a curve, consumers at demand node n choose their consumption level which maximizes their surplus at a given price p_n. We may represent the decision making process of the consumers at the second stage by incorporating, for each demand node n, an optimization problem which maximizes the consumer surplus. Let d_n^* denote the optimal consumption at demand node n such that

$$d_n^* := \arg\max_{d_n \geq 0} \{U_n(d_n) - p_n d_n\}, \tag{3.42}$$

where $U_n(d_n) = \displaystyle\int_0^{d_n} p_n(s)ds$ denotes the willingness to pay function.

Incorporating the consumers' problem will slightly change the equilibrium conditions of the second stage game. Note that $p(\cdot)$ is a decreasing function which leads to $U_n(\cdot)$ being a concave function. Therefore, we can still formulate a convex OPF problem whose solution maximizes the profit of firms, the surplus of consumers, and the profit of TSO in charge of operating the network as indicated in Boucher and Smeers (2001). The modified OPF problem will slightly be different from the ones introduced previously such that its objective functions will also include maximizing $U_n(\cdot)$. For example, the OPF problem involving price-demand curves with deterministic parameters will be formulated, for given x, as

$$Z^*(x) = \min_{\{y,f,d\}} \sum_{g\in G} \sum_{i\in I_g} \sum_{k\in K_g} c_{ik}^g y_{ik}^g - \sum_{n\in N} U(d_n)$$

$$s.t. \sum_{g\in G} \sum_{k\in K_g} y_{jk}^g + f_j \geq d_j \quad (p_j) \qquad \forall j \in \{N \cup I\} \tag{3.43}$$

$$\begin{aligned} d &\geq 0 \\ f &\ \text{satisfy } Cons\_TSO \\ y^g &\ \text{satisfy } Cons\_g(x^g) \qquad\qquad \forall g \in G, \end{aligned}$$

where d_n is the demand of consumers as a reaction of price p_n at node n.

When the parameters in the price-demand curve are assumed to be deterministic, we end up with a two-stage deterministic game in which the first and second stage optimization problems of firm $g \in G$ are still the same. Hence, the arguments in Lemmas 3.3.1 and 3.3.2 still hold. Moreover, $Z^*(x)$ in OPF problem (3.43) is a convex function of x. By using similar arguments in *Proof of Theorem 3.3.4*, one can still formulate a single optimization problem which finds an equilibrium of the corresponding two-stage deterministic game.

Furthermore, the approach in previous sections that assumes a fixed random demand can be extended to a random demand function where key parameters such as reference consumption for a given reference price and elasticities are random. Price elasticities are only known with very little accuracy. It thus makes sense to treat them as uncertain and to embed such parameters in the state of the world. When some parameters in price-demand curves are random, we have a two-stage stochastic game in which the first and second stage optimization problems of firm $g \in G$ in Section 3.3.2 remain the same. Similarly, the arguments in Lemma 3.3.6 and Theorem 3.3.8 still hold. One can write OPF problem (3.43) with explicit dependency on ω and the corresponding optimal value $Z^*(\omega, x)$ will be a convex function of x for every $\omega \in \Omega$. Therefore, $E_\omega[Z^*(\omega, \cdot)]$ is also convex. By using a similar argument in *Proof of Theorem 3.3.7*, one can still formulate a two-stage stochastic program which solves the corresponding two-stage stochastic game.

The standard view in the approach outlined so far is to assume a demand response with a particular functional form and possibly with some random parameters. This idea raises some questions though. The power generation part of the model discussed so far is long-term, in the sense that investments can change the capacity structure of the generation system; therefore, the response of the power system to price changes takes place both through modifications of plant operations and capacities. In contrast, the representation of consumption embedded in a demand function such as (3.42) does not offer that dual long and short term representation. Its most standard interpretation is to assume that it reflects demand-side bidding; that is, participation of the demand to the short-term power market. $p_n(d)$ is typically a short-run response of the demand to price with given capacity in the consuming sector. This creates a model inconsistency between the representation of the supply and demand sectors that can only be removed by introducing a more complex demand model that accounts for both the long run changes of capacity structure in the consuming sector and the short run response of demand with given capacity; for example a representation of consumer decision making including investments in durable energy, using equipment, and habit formation which results in a short run (very low) elasticity, but a longer term adjustment with higher long term elasticity (e.g., Celebi and Fuller (2012)). This discussion goes beyond the scope of this thesis.

3.7 More General Forms of Uncertainties in Spot Markets

In this section, we give a summary of possible extensions to generalize the basic assumption of uncertain demand. In the previous sections, we assume that the demand in the spot market is possibly random. It is also possible to view some of the other parameters as random. These parameters include generation costs $c(\omega)$, transmission capacities $h_l(\omega)$, and power transfer distribution factors $PTDF(\omega)$ with $E_\omega[\| c(\omega) \|] < \infty$, $E_\omega[\| h_l(\omega) \|] < \infty$, and $E_\omega[\| PTDF(\omega) \|] < \infty$, respectively. When these parameters are random, one can prove that the results obtained throughout this chapter still hold as argued briefly in the following propositions.

Proposition 3.7.1. *Let c, h, and $PTDF$ be random and let the second stage problem of each firm $g \in G$ and TSO be feasible and bounded in the neighborhood of a feasible point x^*. Then the equilibrium conditions of the first stage game given in Lemma 3.3.6 (energy-only market), MCP* (3.28) *(energy market with capacity requirements), and Theorem 3.5.3 (energy market with operating-reserve pricing) still hold.*

Proof: The second stage revenues of firm $g \in G$, $\Pi_g^*(\omega, \cdot)$ and $\Pi_g^{*R}(\omega, \cdot)$, are concave functions of x^g for every $\omega \in \Omega$ and $g \in G$ regardless of the random parameter. By similar arguments in *Proofs of Lemma 3.3.6* and *Theorem 3.5.3*, one can derive the equilibrium conditions (3.18), (3.28), and (3.37) for the first stage game. ∎

Proposition 3.7.2. *Let c, h, and $PTDF$ be random and let the corresponding lower level problem* (3.16) *be feasible and bounded in the neighborhood of a feasible point x^*. Then in case of an energy-only market, an energy market with capacity requirements, an energy market with operating-reserve pricing based on observed demand as given in Figure 3.1, or an energy market with demand bidding, the equivalence result with respect to the equilibrium of the two-stage game and the solution of a two-stage stochastic program established under random demand in previous sections (e.g., Theorem 3.3.7) still holds. That is, one can find a perfect competition equilibrium of these markets by solving the corresponding two-stage stochastic program (from the perspective of a central planner).*

Proof: The optimal value of the corresponding OPF problem, $Z^*(\omega, x)$, is a convex function of x for every $\omega \in \Omega$ regardless of the random parameter which implies the convexity of the expected system cost $E_\omega[Z^*(\omega, \cdot)]$ incurred at the second stage. By utilizing Proposition 3.7.1 and using a similar argument in *Proof of Theorem 3.3.7*, the results follow. ∎

3.8 Computational Methods for Solving Two-stage Equilibrium Models

In this section, we discuss the possible computational methods for solving the two-stage games numerically in four different market settings we have considered; namely energy-only markets, energy markets including forward capacity market, energy markets with operating-reserve pricing, and demand bidding.

3.8.1 Solving Two-stage Deterministic Equilibrium Models

MCP Approach

We showed that when we have constant demand, we can always write the equilibrium conditions of the two-stage game as a mixed complementarity problem; for instance the MCP (3.12) in energy-only markets, the MCP consisting of the KKT conditions of (3.27) in energy markets with forward capacity market, the MCP consisting of (3.31) and (3.34) in energy markets with operating-reserve pricing, and finally the MCP consisting of (3.11) and the KKT conditions of (3.43) in energy markets with demand bidding. We can solve these MCPs by using a state of the art MCP solver such as PATH (Dirkse and Ferris (1995b) and Ferris and Munson (2008)). Our results ensure that a point satisfying such an MCP is indeed a solution to the original two-stage game.

(N)LP Approach

We showed in Section 3.3.1.2 that we can find an equilibrium of the two stage game in energy-only markets by solving the linear program (3.14). In Sections 3.4 and 3.5.3.1, we showed that we can extend this result to energy markets with forward capacity market and energy markets with operating-reserve pricing based on an observed demand. Hence instead of solving the MCPs of these two-stage games, we can solve the corresponding (N)LPs (3.14), (3.27), and (3.39) respectively and take the solution as an equilibrium point of the corresponding two-stage game in these market designs. Moreover as mentioned in Section 3.6, under the assumption of demand response we can again formulate a single optimization problem which is a nonlinear program, solve it using an off the shelf nonlinear programming solver, and take its optimal solution as an equilibrium point of the corresponding market. LPs may be much simpler to solve compared to MCPs, especially when we have realistic systems of large networks. Depending on the problem, NLPs may or may not be easier to solve than MCPs.

One should also note that it may not be possible to formulate a single optimization problem for every two-stage game under perfect competition as we elaborate this issue in Section 3.5.3.2 for energy markets with operating-reserve pricing based on installed capacities; in that case, one needs to resort to an MCP approach to provide numerical solutions.

3.8.2 Solving Two-stage Stochastic Equilibrium Models

In general, when we have demand uncertainty, we cannot observe the expected function values explicitly; hence we propose to approximate them by the corresponding sample average functions and solve the resulting approximate problem. The basic method we use is known as *sample-path method* or *sample average approximation method* described briefly in Section 2.4; see for example Robinson (1996) and Shapiro and Homem-De-Mello (1998) for the theoretical background in optimization context. Roughly speaking, in sample-path methods the stochastic problem is observed for a fixed and long sample-path by fixing a large sample size and using the method of common random numbers. Since the sample path and length are fixed, the approximate problem actually becomes a deterministic problem. The resulting deterministic problem is solved by fast and effective solution methods available and its solution is taken as the approximate solution of the stochastic problem. We refer the interested reader to Gürkan et al. (1999a) and Gürkan and Pang (2009) for the theoretical analysis of the sample-path method for solving stochastic equilibrium models.

Next, we explain how we use the basic idea of *sample-path methods* to solve the two-stage stochastic games in each market setting. We again propose two different solution approaches; in the solution approaches outlined below, we use a large, fixed sample size M and an i.i.d. sample point $\omega := \{\omega_1, \omega_2 \ldots, \omega_M\}$. Let $\beta(\omega_1), \beta(\omega_2), \ldots, \beta(\omega_M)$ be the vectors of scarcity rents corresponding to this sample.

MCP Approach

We can always formulate the approximate two-stage stochastic game as a potentially very large MCP. This MCP consists of the KKT conditions of every firm's first and second-stage problems for all realizations in the sample $\{\omega_m\}_{m=1}^M$. For instance, consider the energy-only market in Section 3.3. We cannot observe $E_\omega[\beta_{ik}^{*g}(\omega)]$; however using our random sample of size M, we can approximate it by a sample average function $-\frac{1}{M}\sum_{m=1}^M \beta_{ik}^{*g}(\omega_m)$. It follows from the strong law of large numbers that $-\frac{1}{M}\sum_{m=1}^M \beta_{ik}^{*g}(\omega_m)$ converges to $E_\omega[\beta_{ik}^{*g}(\omega)]$ with probability 1 as M gets large. We can then solve the MCP system which actually consists of (3.44) below and the KKT conditions of OPF problem (3.16) for all $\{\omega_m\}_{m=1}^M$ and take its solution as an approximate solution of the two-stage stochastic game in Section 3.3.2:

$$0 \leq -\frac{1}{M}\sum_{m=1}^{M} \beta_{ik}^{*g}(\omega_m) + \kappa_k \quad \perp \quad x_{ik}^{*g} \geq 0 \quad \forall g \in G, i \in I_g, k \in K_g. \tag{3.44}$$

We can use the same approach for formulating the MCPs of the other three electricity market designs; namely energy markets including forward capacity market, with operating-reserve pricing, and with demand response. Then we can solve the corresponding MCP by a deterministic solver such as PATH. However, this approach has the following drawback: The size of the MCP increases rapidly with the sample size M and the size of the network. Therefore, straightforward construction of an MCP in the stochastic setting for realistic networks and solving this large scale system by using the currently available software may be a computational challenge, as we briefly illustrate in Section 3.9.

Implicit Function Approach

Energy-only Markets: We showed in Section 3.3.2.2 that we can find an equilibrium of the two-stage stochastic game in an energy-only market by solving a two-stage stochastic program. The first and second stages of this two-stage program are formulated in (3.19) and (3.16) respectively. As mentioned earlier, we cannot observe the expected cost function $E_\omega[Z^*(\omega,x)]$ in the first-stage problem (3.19); however we can again approximate it by a sample-average function using the sample $\{\omega_m\}_{m=1}^M$ and solve the approximate problem by using sample-path optimization.

We know that the following holds for the sample-average estimator of the expected cost and its (sub)gradient:

$$\begin{aligned} \frac{1}{M}\sum_{m=1}^{M} Z^*(\omega_m,x) &\quad \to E_\omega[Z^*(\omega,x)] \text{ as } M\to\infty \text{ almost surely for given} x, \text{and} \\ \frac{1}{M}\sum_{m=1}^{M} \beta_{ik}^{*g}(\omega_m) &\quad \to E_\omega[\beta_{ik}^{*g}(\omega)] = \frac{\partial E_\omega[Z^*(\omega,x)]}{\partial x_{ik}^g} \text{ as } M\to\infty \text{ almost surely.} \end{aligned} \tag{3.45}$$

We can construct the approximate problem (3.46) below to approximate (3.19):

$$\min_{x\geq 0} \quad \frac{1}{M}\sum_{m=1}^{M} Z^*(\omega_m,x) + \sum_{g\in G}\sum_{i\in I_g}\sum_{k\in K_g} \kappa_k x_{ik}^g. \tag{3.46}$$

Note that (3.46) is in general an NLP and we can try to solve it by using a standard NLP solver. An efficient NLP solver would require us to provide function values as well as (sub)gradient values of the objective function (3.46) as input. Although we do not have an explicit expression for $Z^*(\omega_m,x)$, given ω_m and x, we can solve the second stage problem (3.16) numerically (by using a standard LP solver) to obtain $Z^*(\omega_m,x)$ at that point. Once the optimal solution of (3.16) is found, we also have $\beta_{ik}^{*g}(\omega_m) \in \frac{\partial Z^*(\omega_m,x)}{\partial x_{ik}^g}$ for every $g,i,$ and k as a by-product. Since x is a right-hand side parameter in (3.16), $\beta_{ik}^{*g}(\omega_m)$ is simply the associated multiplier.

To summarize, this approach would involve solving M consecutive LPs of format (3.16) at any point x that the NLP solver (used for solving (3.46)) would like to evaluate its optimality. The illustration of the corresponding sample-path method is given in Figure 2.2.

Energy Markets with a Forward Capacity Market: In Section 3.4, we showed that the main results of Section 3.3.2.2 can directly be extended to the two-stage game of Section 3.4, namely energy markets with forward capacity market. In this market setting, we can still find an equilibrium of the corresponding two-stage stochastic game by solving the two-stage stochastic program (3.29). By using the fixed sample point $\{\omega_m\}_{m=1}^M$, problem (3.29) can be approximated by (3.47):

$$\begin{aligned} \min_{x \geq 0} \quad & \frac{1}{M}\sum_{m=1}^{M} Z^*(\omega_m, x) + \sum_{g\in G}\sum_{i\in I_g}\sum_{k\in K_g} \kappa_k x_{ik}^g \\ s.t. \quad & \sum_{g,i,k} x_{ik}^g \geq H. \end{aligned} \tag{3.47}$$

The only difference between the approximate problems (3.46) and (3.47) is that a deterministic capacity regulation constraint is imposed in the latter. Hence, the arguments for solving (3.46) are also valid for (3.47).

Energy Markets with Operating-reserve Pricing: In Section 3.5.3, we showed that it is not always possible to formulate a two-stage stochastic program that finds an equilibrium of the two-stage stochastic game under perfect competition. If one can formulate a two-stage program as in Section 3.5.3.1, then it is possible to solve (3.40) in a similar way used for (3.19), as explained in *implicit function approach* for energy-only markets. If one cannot formulate a two-stage program as in Section 3.5.3.2, then in order to find an equilibrium of the corresponding two-stage stochastic game, one has to solve the stochastic complementarity problem (3.37) at the first stage and the optimization problem (3.35) at the second stage.

As explained before, one possible approach is solving the corresponding aggregate MCP system (that is, MCP (3.37) and KKT conditions of (3.35)) to find an approximate equilibrium point of the two-stage stochastic game, as described in MCP approach for stochastic systems. However, the size of the resulting aggregate MCP system grows rapidly with the sample size M. Hence, it may be computationally very time-consuming or impossible to solve such an MCP system for realistic networks.

We next propose another way for finding an approximate equilibrium of the two-stage stochastic game of Section 3.5.3.2 by implicit function approach. First, we approximate (3.37) by using sample average approximations of the expected functions. For given x, one can easily approximate $E_\omega[\beta_{ik}^{*g}(\omega) + rp\_capacity(\frac{ex-ey^*(\omega)}{ex})]$ by the sample average function $-\frac{1}{M}\sum_{m=1}^{M}[\beta_{ik}^{*g}(\omega_m) + rp\_capacity(\frac{ex-ey^*(\omega_m)}{ex})]$. Thus, we construct (3.48) below to approximate (3.37):

$$0 \leq -\frac{1}{M}\sum_{m=1}^{M}[\beta_{ik}^{*g}(\omega_m) + rp\_capacity(\frac{ex^* - ey^*(\omega_m)}{ex^*})] + \kappa_k \perp x_{ik}^{*g} \geq 0, \ \forall g \in G, i \in I_g, k \in K_g. \tag{3.48}$$

As mentioned, we would like to solve (3.48) reminiscent of the implicit function approach. Note that the size of (3.48) does not depend on the sample size M. In order to solve (3.48), we need to provide function and (sub)gradient values of $-\frac{1}{M}\sum_{m=1}^{M}[\beta_{ik}^{*g}(\omega_m) + rp\_capacity(\frac{ex-ey^*(\omega_m)}{ex})]$ at any x that PATH would like to explore. Given any x, we can solve the second stage OPF problem (3.35) M consecutive times and obtain the values of $\beta_{ik}^{*g}(\omega_m)$ and $rp\_capacity(\frac{ex-ey^*(\omega_m)}{ex})$ for each $m \in M$.

Unfortunately, to approximate the (sub)gradient of $E_\omega[\beta_{ik}^{*g}(\omega) + rp\_capacity(\frac{ex-ey^*(\omega)}{ex})]$, we cannot directly use the (sub)gradient values of $\beta_{ik}^{*g}(\omega_m)$. Although $E[\beta_{ik}^{*g}]$ is continuous and $\frac{1}{M}\sum_{m=1}^{M}\beta_{ik}^{*g}(\omega_m)$ becomes almost continuous for large M, $\beta_{ik}^{*g}(\omega_m)$ is actually a piecewise constant function of x. Thus, the (sub)gradient of

$\beta_{ik}^{*g}(\omega_m)$ is either zero or undefined for a given x and ω_m. Hence, we use the following steps to obtain an sample average estimator of $\partial E_\omega[\beta_{ik}^{*g}(\omega)+rp\_capacity(\frac{ex-ey^*(\omega)}{ex})]/\partial x_{ik}^g$ for any $x_{ik}^g > 0$:

- By using (3.31), for any ω with $y_{ik}^*(\omega) > 0$:

$$E_\omega[\beta_{ik}^{*g}(\omega)+rp\_capacity(\tfrac{ex-ey^*(\omega)}{ex})] = E_\omega[(p_i^*(\omega)-c_{ik}^g)].$$

- Again, by (3.31), for any ω with $y_{ik}^*(\omega) = 0$, we have $\beta_{ik}^{*g}(\omega) = 0$. Thus:

$$E_\omega[\beta_{ik}^{*g}(\omega)+rp\_capacity(\tfrac{ex-ey^*(\omega)}{ex})] = E_\omega[rp\_capacity(\tfrac{ex-ey^*(\omega)}{ex})].$$

- Hence, by these two together, we can write:

$$\begin{aligned} E_\omega[\beta_{ik}^{*g}(\omega)+rp\_capacity(\tfrac{ex-ey^*(\omega)}{ex})] = \quad & E_\omega[(p_i^*(\omega)-c_{ik}^g)I_{\{y_{ik}^*(\omega)>0\}}] \\ & +E_\omega[rp\_capacity(\tfrac{ex-ey^*(\omega)}{ex})I_{\{y_{ik}^*(\omega)=0\}}]. \end{aligned} \tag{3.49}$$

By calculating the subgradients of (3.49), we get

$$\begin{aligned} \frac{\partial E_\omega[\beta_{ik}^{*g}(\omega)+rp\_capacity(\frac{ex-ey^*(\omega)}{ex})]}{\partial x_{ik}^g} = \quad & \frac{\partial E_\omega[(p_i^*(\omega)-c_{ik}^g)I_{\{y_{ik}^*(\omega)>0\}}]}{\partial x_{ik}^g} \\ & +\frac{\partial E_\omega[rp\_capacity(\frac{ex-ey^*(\omega)}{ex})I_{\{y_{ik}^*(\omega)=0\}}]}{\partial x_{ik}^g}. \end{aligned} \tag{3.50}$$

- Note that for a fixed sample $\{\omega_m\}_{m=1}^M$, we can always perturb $x_{ik}^g > 0$ small enough such that $x_{ik}^g + \Delta x_{ik}^g > 0$ and the values of the indicator functions $I_{\{y_{ik}^*(\omega)>0\}}$ and $I_{\{y_{ik}^*(\omega)=0\}}$ do not change; hence they can be treated as constants.

- Next, we approximate the (sub)gradients of $E_\omega[(p_i^*(\omega) - c_{ik}^g)I_{\{y_{ik}^*(\omega)>0\}}]$ and $E_\omega[rp\_capacity(\frac{ex-ey^*(\omega)}{ex})I_{\{y_{ik}^*(\omega)=0\}}]$ for any $x_{ik}^g > 0$ by using the following sample-average estimators. Let $u(\omega_m) = \dfrac{ex-ey^*(\omega_m)}{ex}$, then:

$$\frac{1}{M}\sum_{m=1}^{M}\frac{\partial p_i^*(\omega_m)}{\partial x_{ik}^g} I_{\{y_{ik}^*(\omega_m)>0\}} \to \frac{\partial E_\omega[(p_i^*(\omega)-c_{ik}^g) I_{\{y_{ik}^*(\omega)>0\}}]}{\partial x_{ik}^g}, \text{ and}$$

$$\frac{1}{M}\sum_{m=1}^{M}\frac{\partial rp\_capacity(u(\omega_m))}{\partial u(\omega_m)}\frac{\partial u(\omega_m)}{\partial x_{ik}^g} I_{\{y_{ik}^*(\omega_m)=0\}} \to \frac{\partial E_\omega[rp\_capacity(\frac{ex-ey^*(\omega)}{ex}) I_{\{y_{ik}^*(\omega)=0\}}]}{\partial x_{ik}^g} \quad (3.51)$$

as $M \to \infty$ almost surely.

Given any x, we can calculate the the corresponding (sub)gradient values by using (3.50) and (3.51). Thus, one needs to calculate the values $(y^*(\omega_m), u(\omega_m))$ and the subgradients $\partial rp\_capacity(u(\omega_m))/\partial u(\omega_m)$, $\partial u(\omega_m)/\partial x_{ik}^g$, and $\partial p_i^*(\omega_m)/\partial x_{ik}^g$. Once the function $rp\_capacity(\cdot)$ is explicitly defined, all of these values, except subgradient of $p_i^*(\omega_m)$, are straightforward to calculate from the solution of OPF problem (3.35). For instance,

$$(u(\omega_m), \frac{\partial u(\omega_m)}{\partial x_{ik}^g}) = (\frac{ex-ey^*(\omega_m)}{ex}, \frac{ey^*(\omega_m)}{ex^2}).$$

For the calculation of the subgradient of $p_i^*(\omega_m)$, Castillo et al. (2006) gives an integrated approach which at once yields all the sensitivities of the optimal solution of an NLP problem to changes in the parameter values. They illustrate how to obtain the directional and partial derivatives of the optimal objective function value, optimal primal, and dual variable values with respect to the parameters of a general NLP problem by a single calculation. Once we solve the OPF problem (3.35), we utilize the approach of Castillo et al. (2006) to calculate the subgradient of $p_i^*(\omega_m)$.

Energy Markets with Demand-side bidding: Similar to the energy-only market, the first stage problem of the two-stage stochastic model can be approximated by (3.46). The only difference between the approximate problems of energy-only market and energy market with demand response is the second stage OPF problem. The second stage OPF problem of energy market with demand response would be almost identical to (3.43) with explicit depends on ω_m. Hence, the discussion related to solving (3.46) remain applicable in this case as well.

3.9 Numerical Illustration

The numerical experiments reported here have been conducted to serve two purposes. Firstly, we would like to compare the performance of the computational methods discussed in Section 3.8 and secondly we would like to see the impact of the market designs discussed in this chapter on investment incentives of the firms. To this end, we apply in this section the methods proposed in Section 3.8 to solve the equilibrium for the electricity markets discussed throughout this chapter under both deterministic and stochastic setting.

All the numerical experiments reported are performed by a Dell PC with Dual-Intel Xeon, 575 2.66 GHz processors, and 2 GB 266 MHz DDR Non-ECC SDRAM Memory using 576 Windows 2000. The solvers utilized for each problem depend on both the type of application problem and the method, which are summarized in Tables 3.1 and 3.2. Note that the methodologies discussed in Section 3.8 allow modularity; that is, one can always use any off-the-shelf solvers. In Table 3.2, (Non)Linear Program ((N)LP) and Mixed Linear/Nonlinear Complementarity Problem (MLCP/MNCP) are used to solve first and second stage problems simultaneously whereas Stochastic Program (SP) and MCP s.t. NLP are based on implicit function approach discussed in Section 3.8; that is, first stage problem and second stage problems at each realization are solved iteratively. Thus, one can use a separate solver for each stage. In our case, we use CONOPT(warm start) and SNOPT sequentially for SPs and PATH for MCPs to find an equilibrium of the first stage. These first stage solvers call the second stage solver M consecutive times at a point x to explore its optimality. The up-to-date information on the first stage solvers can be found in the online documentations available by GAMS (see GAMS (2012)). In addition, for solving the second stage OPF problem we use the deterministic nonlinear optimization code E04UCC of NAG C library, Mark 7, NAG (2002). E04UCC is designed to minimize an arbitrary smooth function subject to constraints, which may include simple bounds on the variables, linear constraints, and smooth nonlinear constraints. Essentially, it is a sequential quadratic programming method incorporating an augmented Lagrangian merit function and a BFGS quasi-Newton approximation to the Hessian of the Lagrangian.

Application Problem	**Remedy Mechanism**
EO	Energy-only with VOLL pricing
ECAP	Capacity markets
EORP1	Operating-reserve pricing based on demand
EORP2	Operating-reserve pricing based on capacity

Table 3.1: Overview of application problems

We consider a competitive power market of six nodes given by Chao and Peck (1998) in Figure 3.3, which has the following characteristics:

- The nodes 1, 2, and 4 are supply nodes ($I := \{1,2,4\}$) and the nodes 3, 5, and 6 are demand nodes ($N := \{3,5,6\}$).

- Without loss of generality, at each supply node there is a single firm investing in one technology

Method	**Application Problem**	**Solver First Stage**	**Solver Second Stage**
LP/NLP	All except EORP2	CPLEX/SNOPT(GAMS) (1^{st} and 2^{nd} stage)	with 1^{st} stage
MLCP/MNCP	All	PATH(GAMS) (1^{st} and 2^{nd} stage)	with 1^{st} stage
SP	All except EORP2	CONOPT(warm start) and SNOPT(GAMS)	E04UCC (NAG C)
MNCP s.t. NLP	EORP2	PATH(GAMS)	E04UCC (NAG C)

Table 3.2: Overview of solvers utilized for each method and problem type

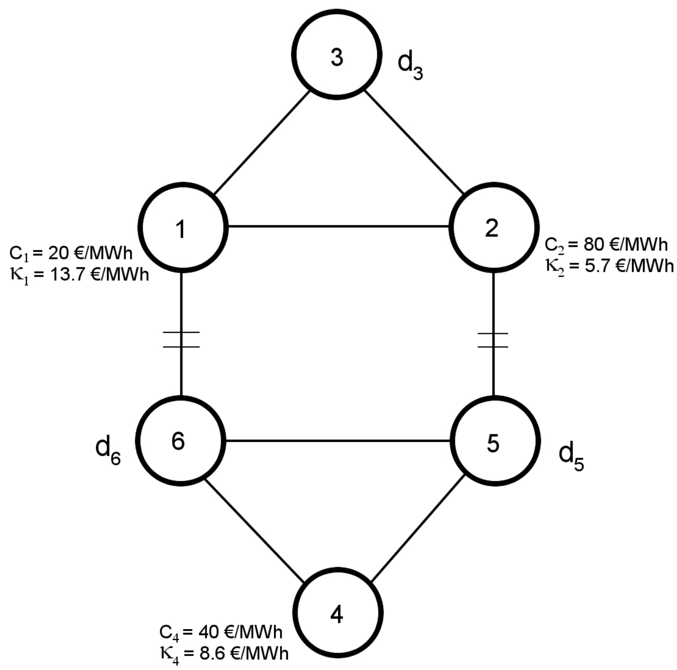
$(g = i = k)$. The unit generation and investment costs of these firms are given in Figure 3.3. The corresponding data for marginal costs for these three generator types are taken from Schulkin et al. (2010).

– The demand in nodes 3, 5, and 6 are uniformly distributed with corresponding upper (D^{max}) and lower bounds (D^{min}) given in Figure 3.3.

– To analyze the electricity market with limited network capacity, we assume all lines have infinite capacities except for the lines (1-6) and (2-5) where $h_{16} = 8$ and $h_{25} = 8$. All line impedances are equal to 1 except for (1-6) and (2-5) that have impedances equal to 2. The PTDFs in Table 3.3 indicate the flows through lines (1-6) and (2-5) resulting from one unit injection at each node of the network and withdrawal at node 6, which is taken as the hub.

– VOLL is taken as 10,000 Euro/MWh which is a standard value of lost load used in the literature (e.g., Stoft (2002), Hogan (2005)).

– In case of operating-reserve pricing, both operating-reserve price curves introduced in Sections 3.5.3.1 and 3.5.3.2 are assumed to be smooth sigmoid functions as given in Figures 3.1 and 3.2, respectively. The mathematical representation of a sigmoid function is: $F(t) = \frac{F_{max}}{1+e^{k(t-F_{mid})}}$ where F_{max} is the maximum value of the function. F_{mid} gives the mid value of the function ($F(F_{mid}) = F_{max}/2$). Finally, k gives the curvature information centered on F_{mid}. Regarding the approximating function of the operating-reserve price curve offered by Hogan (2005) in Figure 3.1 and the operating-reserve price curve based on installed capacity in Figure 3.2, one should take $F_{max} = 10,000$ and $F_{mid} = 0.05$. In order to determine the value of k, the error between the real function and the approximating function is calculated for different values of k and $k = 133$ gives the best approximation with the smallest error. Therefore for the numerical results reported in this section, we use the formulation $rp(f(x,y,d)) = \dfrac{10000}{1+e^{133(f(x,y,d)-0.05)}}$ for the operating-reserve price curves, $rp\_demand(\frac{ex-ey^*}{ed})$ and $rp\_capacity(\frac{ex-ey^*}{ex})$, given in Figures 3.1 and 3.2, respectively. The unit of operating-reserve price in these figures are taken as Euro/MWh.



n	$D^{min}(GW)$	$D^{max}(GW)$
3	8	12
5	3	5
6	15	20

Figure 3.3: The data of 6-node example

Next, we solve two-stage deterministic and two-stage stochastic games by using the approaches proposed in Section 3.8.

Example 3.9.1. Perfect competition equilibria under deterministic demand

We first consider the situation in which the firms solve their first stage problem by simply taking the expected values of the demand: $d_3 = 10, d_5 = 4$, and $d_6 = 17.5$. Then the corresponding game is a two-stage deterministic game. In the energy-only (EO) market, energy market with forward capacity market

Node	Line (1-6)	Line (2-5)
1	0.6250	0.3750
2	0.5000	0.5000
3	0.5625	0.4375
4	0.0625	-0.0625
5	0.1250	-0.1250

Table 3.3: Power distribution factors ($PTDF_{l,j}$)

(ECAP), and energy market with operating-reserve pricing based on the observed demand (EORP1), we find the equilibrium of the two-stage deterministic game by utilizing both MCP and (N)LP approaches. In the energy market with operating-reserve pricing based on installed capacity (EORP2), we find the equilibrium of the two-stage deterministic game by utilizing MCP approach. In the ECAP market, we impose that the capacity market requires 11.2% more capacity than the total expected demand ($H = 35.2$) which results in same reserve capacity as in EORP2 market. The results are presented in Tables 3.4 and 3.5. Furthermore, the results obtained by MCP approach and (N)LP approach for EO, ECAP, and EORP1 markets are identical and the computation time for both approaches takes less than a second.

Investment Incentives: Regarding the EO market result as reference, Tables 3.4 and 3.5 indicate that the investments increase when there is capacity market or operating-reserve pricing. The prices in both EORP markets remain identical whereas the prices in ECAP market are lower. However, there is an extra capacity price $\lambda^* = 5.7$ euro/MWh paid to the firms in ECAP market. If the consumers are paying this capacity price, then one can conclude that ECAP market results in identical prices as well.

Compared to EO market, total generation capacity investments are higher in ECAP and EORP markets; hence the system has higher investment costs in these markets. Regarding the comparison between different market designs, investment incentives are higher in EORP2 market than in EORP1 market. In addition, ECAP market results in identical investment levels as in EORP2 market since H is chosen to be equal to the total generation capacity in EORP2 market. If H were chosen to be equal to the total generation capacity in EORP1 market, then ECAP market would result in identical investment levels as in EORP1 market. Based on our extensive numerical experiments, we conjecture that for any operating-reserve function rp in EORP1 or EORP2 market, there is a corresponding H in ECAP market which results in identical total reserve capacity and mix of technologies. This observation is also made by Hobbs et al. (2001). Furthermore, if the "correct" extra capacity price ($\lambda^* = 5.7$) is paid by the consumers, then one can conclude that when H is equal to the total generation capacity in an EORP market, there is no difference between ECAP and the corresponding EORP market.

Market Design	GW x_1^*	x_2^*	x_4^*	euro / MWh p_3^*	p_5^*	p_6^*	Investment and Operational Cost (k Euro)
EO	31.5	0	0	33.7	33.7	33.7	1061.5
EORP1	31.5	3.3	0	33.7	33.7	33.7	1080.4
EORP2	31.5	3.7	0	33.7	33.7	33.7	1082.6
ECAP	31.5	3.7	0	28.0	28.0	28.0	1082.6

Table 3.4: Equilibrium in two-stage deterministic game with infinite transmission line capacities

Market Design	GW x_1^*	x_2^*	x_4^*	euro / MWh p_3^*	p_5^*	p_6^*	Investment and Operational Cost (k Euro)
EO	21.6	0	9.9	35.4	46.9	50.3	1208.9
EORP1	21.6	3.3	9.9	35.4	46.9	50.3	1227.9
EORP2	21.6	3.7	9.9	35.4	46.9	50.3	1230.1
ECAP	21.6	3.7	9.9	29.7	41.2	44.6	1230.0

Table 3.5: Equilibrium in two-stage deterministic game with limited network capacity

Example 3.9.2. Perfect competition equilibria under demand uncertainty

We now consider the situation in which the firms take the randomness of demand into account and the corresponding two-stage stochastic game needs to be solved. We first solve the two-stage stochastic games in EO, ECAP, and EORP2 markets by using both MCP and implicit function approaches discussed in Section 3.8 and compare the performance of these approaches. We use sample sizes of $M = 1000$ and $M = 8760$ and compare the computational performance of these two approaches in the energy-only market in Table 3.6, in EORP2 market in Table 3.7, and in ECAP market in Table 3.8, respectively. We report optimal capacities installed together with the solution time. We skip the comparison of the computational performance of these approaches for EORP1 market since the corresponding solution times are similar to that of EO market. Note that the particular problem being solved in the MCP approach is referred as mixed linear complementarity problem (MLCP) for energy-only and ECAP markets and mixed nonlinear complementarity problem (MNCP) for EORP2 market. Moreover, the particular problem being solved in the implicit function approach is referred as stochastic program (SP) for energy-only and ECAP markets and MNLCP s.t. NLP in EOPR2 market (since mixed nonlinear complementarity problem is solved at the upper level subject to the optimal solution set of the nonlinear program at the lower level). Next for $M = 8760$, we compare the corresponding generation capacity investments, average consumer prices, and average investment and operational (generation and curtailment) costs for EO, ECAP, and EORP markets for both unlimited and limited network in Tables 3.10 and 3.11, respectively.

Computational Performance: The comparisons between the MCP approach (MLCP or MNCP) and implicit function approach (SP or MNCP s.t. NLP) in all tables indicate that the computational time to solve the two-stage stochastic game by MCP approach increases rapidly with the sample size M. Since the implicit function approach involves solving the lower level (N)LP problem M consecutive times, the computational time increases with sample size M in a linear way. Hence, the MCP approach is more efficient in solving problems with smaller sample size (e.g., $M = 1000$) and implicit function approach is more efficient in solving problems with large sample size (e.g., $M = 8760$). We note that the solutions of the approximating problem with small sample size ($M = 1000$) are almost identical to the solutions of the approximating problem with large sample size ($M = 8760$) in Tables 3.6-3.8. This is mainly a consequence of using uniform distribution in which case the approximation functions converge to their limit very rapidly. Next, we consider an example in the energy-only market in which the stochastic demand in nodes $3, 5$, and 6 has triangular distribution with peak values $9, 3.5$, and 16, respectively. Table 3.9 contains the equilibrium points of this two-stage game. We see that the equilibria for different sample sizes show wider variations in this case. Therefore, depending on the variability of the underlying random data and the final accuracy desired, one may sometimes prefer to solve the two-stage game for larger sample sizes (e.g., $M = 8760$) which get closer to the true solution.

Investment Incentives: The comparison of investment capacities installed by each technology, average consumer prices, and total investment and operational cost of the system under each market design is given in Tables 3.10 and 3.11. The values in these tables represent the computations done with sample size of 8760. The results indicate that the total generation capacity in energy-only market is 1-2% less than the total peak demand, whereas ECAP and operating-reserve pricing in EORP1 and EORP2 markets result in total generation capacities that are about 4.5% - 5.0% above the total peak demand level. Thus, energy-only markets with VOLL pricing tend to lead to total generation capacity below the peak load with a certain probability whereas energy markets with a forward capacity market or operating-reserve pricing result in higher investments. VOLL pricing (with high values of VOLL) may also result in similar investment levels as the other market designs if forced outages are taken into account and the demand is assumed to have a distribution without a finite support (see the result of Hobbs et al. (2001)). In our experiments, since we do not consider forced outages and we assume demand distributions with finite support, the regulator's reserve target assumptions (in case of capacity markets or operating-reserve pricing) result in total generation capacity higher than the peak demand level which cannot be achieved by VOLL pricing.

As observed in the deterministic case, EORP2 market results in higher investment level of peak unit compared to EORP1 market. Different from the deterministic case, the average consumer prices in EORP2 market are slightly higher than the average prices in EORP1 market. In ECAP market, we used the total capacity requirement $H = 38.8$. As one can see in Tables 3.10 and 3.11, the investment levels obtained in ECAP market are close to the investment levels in EORP markets. Similar to the deterministic case, we conjecture a similar result of Hobbs et al. (2001) that for any operating-reserve function rp in EORP1 or EORP2 market, there is a corresponding H in ECAP market which results in identical total reserve capacity and mix of technologies.

Although the average prices in ECAP market are the lowest, there is again an extra capacity price $\lambda^* = 5.7$ euro/MWh which is likely be paid by the consumers. Furthermore compared to the EO market, the relative increase in system operational and investment cost in ECAP and EORP markets under uncertainty is lower than the increase in the total cost for the deterministic case. This can be explained by the impact of higher operational cost due to the curtailment at some realizations and lower investment cost in EO market versus lower operational and higher investment costs in the other market designs. Finally, the comparison of the results in deterministic and stochastic settings indicate that when uncertainty of future demand is taken into account by the risk neutral generators, their investments result in higher total generation capacity and a broader mix of technologies compared to the case when generators invest based on the expected demand.

Sample size M	Infinite network capacity (x_1^*, x_2^*, x_4^*) *CPU time*		Limited network capacity (x_1^*, x_2^*, x_4^*) *CPU time*	
	MLCP	SP	MLCP	SP
1000	(32.9, 2.0, 1.5) 00h 00’47”	(32.9, 2.0, 1.5) 00h 13’46”	(22.1, 1.3, 13.0) 00h 05’04”	(22.1, 1.3, 13.0) 00h 27’45’
8760	(32.9, 2.1, 1.5) 06h 16’ 49”	(32.9, 2.1, 1.5) 01h 36’ 25”	(22.0, 1.2, 13.3) 23h 01’ 09”	(22.0, 1.2, 13.3) 02h 32’ 38”

Table 3.6: Equilibrium of two-stage stochastic game in energy-only markets (EO)

Sample size M	Infinite network capacity (x_1^*, x_2^*, x_4^*) *CPU time*		Limited network capacity (x_1^*, x_2^*, x_4^*) *CPU time*	
	MNCP	MNCP s.t. NLP	MNCP	MNCP s.t. NLP
1000	(32.9, 4.4, 1.5) 00h 01’05”	(32.3, 3.6, 2.9) 01h 0’07”	(22.1, 3.7, 12.9) 00h 05’04”	(22.1, 3.7, 12.9) 01h 15’53”
8760	(32.9, 4.5, 1.5) 10h 46’ 22”	(32.9, 4.5, 1.5) 09h 31’ 38”	NA (time limit) 75h 48’ 19”	(22.0, 3.8, 13.0) 08h 19’ 42”

Table 3.7: Equilibrium of two-stage stochastic game in energy markets with operating-reserve price (EORP2)

3.10 Conclusions

We have considered alternative market designs which may remedy the resource adequacy problem in restructured electricity markets. Each market design corresponds to a different type of multi-agent model formulation depending on the remedy mechanism and the assumption on the market agents’ behaviors. Taking into account the uncertainty or variability of parameters in these multi-agent models may lead

Sample size M	Infinite network capacity (x_1^*, x_2^*, x_4^*) *CPU time*		Limited network capacity (x_1^*, x_2^*, x_4^*) *CPU time*	
	MNCP	MNCP s.t. NLP	MNCP	MNCP s.t. NLP
1000	(32.9, 4.4, 1.5) 00h 00’33”	(32.9, 4.4, 1.5) 00h 27’48”	(22.1, 3.7, 12.9) 00h 02’21”	(22.1, 3.7, 12.9) 00h 24’50”
8760	(32.9, 4.4, 1.5) 4h 44’ 11”	(32.9, 4.4, 1.5) 01h 54’ 11”	(22.1,3.6,13.1) 21h 38’ 34”	(22.1, 3.6, 13.1) 01h 54’ 49”

Table 3.8: Equilibrium of two-stage stochastic game in energy markets with forward capacity requirements (ECAP)

Sample size M	Infinite network capacity (x_1^*, x_2^*, x_4^*) *CPU time*		Limited network capacity (x_1^*, x_2^*, x_4^*) *CPU time*	
	MLCP	SP	MLCP	SP
1000	(31.5, 2.2, 1.1) 00h 00’51”	(31.5, 2.2, 1.1) 00h 14’18”	(21.5, 1.0, 12.3) 00h 02’21”	(21.5, 1.0, 12.3) 00h 23’48’
8760	(31.5, 2.4, 1.2) 10h 43’ 53”	(31.5, 2.4, 1.2) 02h 06’ 01”	(21.5, 0.8, 12.8) 34h 18’ 31”	(21.5, 0.8, 12.8) 04h 59’ 28”

Table 3.9: Equilibrium of two-stage stochastic game in EO markets with stochastic demand sampled from triangular distribution

	GW			euro / MWh			Average Investment and
Market Design	x_1^*	x_2^*	x_4^*	$E[p_3^*]$	$E[p_5^*]$	$E[p_6^*]$	Operational Cost (k Euro)
EO	32.9	2.1	1.5	33.6	33.6	33.6	1113.6
EORP1	32.9	4.3	1.5	33.5	33.5	33.5	1125.7
EORP2	32.9	4.5	1.5	33.8	33.8	33.8	1126.8
ECAP	32.9	4.4	1.5	28.0	28.0	28.0	1126.3

Table 3.10: Equilibrium in two-stage stochastic game with infinite transmission line capacities

	GW			euro / MWh			Average Investment and
Market Design	x_1^*	x_2^*	x_4^*	$E[p_3^*]$	$E[p_5^*]$	$E[p_6^*]$	Operational Cost (k Euro)
EO	22.0	1.2	13.3	36.0	46.5	49.5	1256.8
EORP1	22.1	3.5	13.1	35.4	46.6	49.8	1268.6
EORP2	22.0	3.8	13.0	38.5	50.2	53.6	1269.3
ECAP	22.1	3.6	13.1	29.8	41.0	44.3	1269.1

Table 3.11: Equilibrium in two-stage stochastic game with limited network capacity

to large-scale problems which are computationally complex to solve due to scarcity of resources (e.g., available memory and speed of computers). We show that, in perfectly competitive markets, most of the market models can be cast into deterministic or stochastic optimization problems similar to the early capacity expansion models of a regulated monopoly. This result also suggests that an equilibrium of a single-stage (open loop) model in which investment and operation decisions are made simultaneously coincides with an equilibrium of a two-stage (closed loop) model where investment and operation decisions are made sequentially. By using this result, we show that we can utilize sample-path methods together with the powerful available solvers for deterministic optimization problems, which provides computational simplicity for solving such models of realistic systems with stochastic elements.

By utilizing numerical experiments, we also provide insights on the impact of demand uncertainty and to what extend these market designs provide incentives to invest in generation capacities. Firstly, uncertainty of demand leads to higher investments in total generation capacity and a broader mix of technologies compared to the investment decisions assuming average demand levels. Furthermore in energy-only markets, peak-load generators tend to under-invest and curtail peak load so that they receive positive margin via high prices (e.g., VOLL) to cover their long-run marginal costs. In energy markets with a forward capacity market or with operating-reserve pricing, peak-load generators receive positive margin not only via curtailment but also by providing more capacity to the system. Therefore for the same VOLL (or price cap) level, energy-only markets with VOLL pricing tend to lead to total generation capacity below the peak load with a certain probability whereas energy markets with a forward capacity market or operating-reserve pricing result in higher investments under the assumptions of random demand with finite support and no forced outages. Moreover given similar regulator targets, operating-reserve pricing based on installed capacity provides higher incentives than operating-reserve pricing based on observed demand and it does not increase the total investment and operational cost in the system significantly. Lastly, the regulator decisions (e.g., reserve capacity target) in capacity markets and operating-reserve pricing can be chosen in such a way that results in very similar investment levels and fuel mix of generation capacities in both market designs.

Finally, the result of the prevalence of stochastic programming for providing solutions to stochastic equilibrium models can be extended to generation capacity investment strategies in perfectly competitive electricity markets with different regulatory mechanisms such as emission trading scheme (Gürkan et al. (2012)) or renewable obligations (Gürkan and Langestraat (2012)). In addition, risk aversion can also be included by using “coherent risk measures” in the first stage problem of the firms. This allows assessment of investment incentives of risk averse generators by utilizing a two-stage stochastic program when the markets are perfectly competitive and “complete” (Ralph and Smeers (2011)). However, this does not necessarily guarantee that every equilibrium model under perfect competition can be cast as two-stage stochastic program since equilibrium problems are indeed broader than optimization problems. The natural approach is to resort initially to complementarity formulations as to model competitive electricity markets. Depending on the structure of the market and regulatory intervention, market equilibrium may or may not be equivalent to the solution of a system optimization. For instance in one case of operating-reserve pricing, we obtain an equilibrium problem that is not equivalent to an optimization problem.

3.A Proofs of Lemmas and Theorems

Proof of Lemma 3.3.3. The first stage problem (3.10) is concave in x^g and the second stage problem (3.9) is a linear program and therefore is convex. Hence, (3.11) and (3.8) are necessary and sufficient optimality conditions for all firms at both stages. By using the result, $x^* = y^*$, from Lemma 3.3.1, we can rewrite (3.8) by replacing y_{ik}^{*g} with x_{ik}^{*g}. Thus, the first two complementarity equations in (3.8) reduce to:

$$\begin{array}{llll} (i) & 0 \leq c_{ik}^g - p_i^* + \beta_{ik}^{g*} \quad \perp \quad x_{ik}^{*g} \geq 0 & \forall g \in G, i \in I_g, k \in K_g \\ (ii) & \beta_{ik}^{g*} \geq 0 & \forall g \in G, i \in I_g, k \in K_g. \end{array}$$

In addition by Lemma 3.3.2,

$$(iii) \quad 0 \leq -\beta_{ik}^{*g} + \kappa_k \quad \perp \quad x_{ik}^{*g} \geq 0 \quad \forall g \in G, i \in I_g, k \in K_g$$

should hold at the first stage. As a result, a solution x^* which satisfies *(i)-(iii)* would satisfy the following complementarity conditions as well:

$$0 \leq c_{ik}^g - p_i^* + \kappa_k \quad \perp \quad x_{ik}^{*g} \geq 0 \quad \forall g \in G, i \in I_g, k \in K_g.$$

On the other hand, let x^* be a solution to the complementarity conditions in (3.12). Then it is an equilibrium of the two-stage game where $\beta_{ik}^{*g} = p_i^* - c_{ik}^g = \kappa_k$ for $x_{ik}^{*g} > 0$ and $0 \leq \beta_{ik}^{*g} \leq \kappa_k$ for $x_{ik}^{*g} = 0$.∎

Proof of Lemma 3.3.6. Similar to *Proof of Lemma 3.3.2,* first stage problem of each firm $g \in G$ can be formulated as

$$\max_{x^g \geq 0} \quad E_\omega[\Pi_g^*(\omega, x^g)] - \sum_{i \in I_g} \sum_{k \in K_g} \kappa_k x_{ik}^g, \tag{A1}$$

where $\Pi_g^*(\omega, x^g)$ is the optimal value of the firm g's second stage problem (3.15) at realization $\omega \in \Omega$. Similar to the arguments in *Proof of Lemma 3.3.2,* $\Pi_g^*(\omega, x^g)$ is a concave function of x^g for all fixed $\omega \in \Omega$; hence the expectation $E_\omega[\Pi_g^*(\omega, x^g)]$ is a concave function of x^g as well. By using the fact that (3.15) is a linear program, we know that at any given x^g if $\Pi_g^*(\omega, x^g)$ is finite then $\beta_{ik}^{*g}(\omega)$ is a subgradient of $\Pi_g^*(\omega, \cdot)$. Moreover, $\beta_{ik}^{*g}(\omega)$ is unique and $\Pi_g^*(\omega, \cdot)$ is differentiable except on a set L of Lebesgue measure zero. Thus,

$$\frac{\partial \Pi_g^*(\omega, \cdot)}{\partial x_{ik}^g} = \beta_{ik}^{*g}(\omega) \text{ except on } L.$$

Since $\beta_{ik}^{*g}(\omega)$ is a subgradient of $\Pi_g^*(\omega, \cdot)$, $E_\omega[\beta_{ik}^{*g}(\omega)]$ is a subgradient of $E_\omega[\Pi_g^*(\omega, \cdot)]$. Moreover $d(\omega)$ is a continuous random variable; hence L has a probability measure zero. Therefore, $E_\omega[\beta_{ik}^{*g}(\omega)]$ is unique and

$$\frac{\partial E_\omega[\Pi_g^*(\omega,\cdot)]}{\partial x_{ik}^g} = E_\omega[\beta_{ik}^{*g}(\omega)]. \tag{A3}$$

As a result, we can conclude that $E_\omega[\Pi_g^*(\omega,\cdot)]$ is concave and differentiable. Hence, x^{*g} is optimal for firm g's problem (3.17) if and only if it satisfies the optimality conditions of problem (A1) given as

$$0 \leq -\frac{\partial E_\omega[\Pi_g^*(\omega,x^{*g})]}{\partial x_{ik}^g} + \kappa_k \quad \perp \quad x_{ik}^{*g} \geq 0 \quad \forall i \in I_g, k \in K_g.$$

By using the equality in (A3), we get the following equilibrium conditions for the first-stage game:

$$0 \leq -E_\omega[\beta_{ik}^{*g}(\omega)] + \kappa_k \quad \perp \quad x_{ik}^{*g} \geq 0 \quad \forall g \in G, i \in I_g, k \in K_g. \blacksquare$$

Proof of Theorem 3.3.8. Both (3.23) and (3.24) are convex problems. The necessary and sufficient KKT optimality conditions of (3.23) and (3.24) for all firms and TSO together with the market clearing conditions can be written as

$$0 \leq -\sum_{m=1}^{M} \pi_m \beta_{ik}^{*g}(\omega_m) + \kappa_k \quad \perp \quad x_{ik}^{*g} \geq 0 \quad \forall g \in G, i \in I_g, k \in K_g; \tag{A4}$$

where for $m = 1,2,\ldots,M$:

$(y^*(\omega_m), \beta^*(\omega_m), p^*(\omega_m))$ satisfy $MCP\_Firms(\omega_m, x)$

$(f^*(\omega_m), \rho^*(\omega_m), \lambda_l^{*+}(\omega_m), \lambda_l^{*-}(\omega_m))$ satisfy $MCP\_TSO(\omega_m)$

$(y^*(\omega_m), \delta^*(\omega_m), f^*(\omega_m), p^*(\omega_m))$ satisfy $MCP\_Market(\omega_m)$.

In (A4), the first line is equivalent to the equilibrium conditions of the first stage game given in (3.22) and the rest is KKT optimality conditions of the OPF problem (3.16) for every ω_m. Hence, a solution to the MCP (A4) is an equilibrium of the corresponding two-stage stochastic game with finite number of demand scenarios. Furthermore, since (A4) consists of necessary and sufficient optimality conditions of every firm's problem in both two-stage and single-stage games, an equilibrium of the two-stage stochastic game, if it exists, is also an equilibrium of the single-stage stochastic game.$\blacksquare$

Proof of Theorem 3.5.2. The second stage problem (3.30) of each firm $g \in G$ is a linear program where x^g appears both as a coefficient in the objective function and as a the right side parameter in the constraints. Since the objective function is a concave function of x^g and the corresponding constraints are convex in x^g, the optimal objective function value, $\Pi_g^{*R}(x^g)$, is a concave function of x^g in (3.30). When $\Pi_g^{*R}(x^g)$ is finite in the neighborhood of x^{*g}, it is also subdifferentiable at x^{*g} and $(\beta_{ik}^{*g} + \gamma^*)$ is a subgradient of $\Pi_g^{*R}(x^{*g})$. Hence, x^{*g} is an optimal solution of (3.33) for each firm $g \in G$ if and only if there exists

$(\beta_{ik}^{*g}+\gamma^*) \in \frac{\partial \Pi_g^{*R}(x^{*g})}{\partial x_{ik}^g}$ satisfying the necessary and sufficient optimality conditions

$$0 \le -(\beta_{ik}^{*g}+\gamma^*)+\kappa_k \quad \perp \quad x_{ik}^{*g} \ge 0 \quad \forall g \in G, i \in I_g, k \in K_g.$$

After plugging in $\gamma^* = rp(ex^*-ey^*, x^*, d)$, we get the equilibrium conditions in first stage game as

$$0 \le -\beta_{ik}^{*g} - rp(ex^*-ey^*, x^*, d)+\kappa_k \quad \perp \quad x_{ik}^{*g} \ge 0 \quad \forall g \in G, i \in I_g, k \in K_g. \blacksquare$$

Proof of Theorem 3.5.3. $\Pi_g^{*R}(\omega, x^g)$ is the optimal objective function value of the second stage problem (3.30) in stochastic setting defined for each $\omega \in \Omega$. As argued in *Proof of Theorem 3.5.2*, $\Pi_g^{*R}(\omega, x^g)$ is a concave function of x^g; hence $E_\omega[\Pi_g^{*R}(\omega,\cdot)]$ is concave. By using an argument similar to the one in *Proof of Lemma 3.3.6*, we can conclude that $E_\omega[\Pi_g^{*R}(\omega,\cdot)]$ is differentiable and

$$\frac{\partial E_\omega[\Pi_g^{*R}(\omega,\cdot)]}{\partial x_{ik}^g} = E_\omega[\beta_{ik}^{*g}(\omega)+\gamma^*(\omega)].$$

Hence, x^{*g} is an optimal solution of (3.36) for each firm $g \in G$ if and only if it satisfies the optimality conditions

$$0 \le -E_\omega[\beta_{ik}^{*g}(\omega)+\gamma^*(\omega)]+\kappa_k \quad \perp \quad x_{ik}^{*g} \ge 0 \quad \forall g \in G, i \in I_g, k \in K_g.$$

After plugging in $\gamma^*(\omega) = rp(ex^*-y^*(\omega), x^*, d(\omega))$, we get the equilibrium conditions in (3.37). $\blacksquare$

Proof of Remark 3.5.4. $(x,y) \in \Re_+^{M\times M}$ where $M := G \times I \times K$. $R\_demand(\frac{ex-ey}{ed})$ is concave in (x,y) iff its Hessian $H := \Re^{2M\times 2M}$ is negative semidefinite. Next we calculate the Hessian of $R\_demand(\frac{ex-ey}{ed})$: Let $H^1 := \Re^{M\times M}$, then

$$\nabla^2_{xx} R\_demand(\frac{ex-ey}{ed}) = \nabla^2_{yy} R\_demand(\frac{ex-ey}{ed}) = H^1 \text{ and}$$

$$\nabla^2_{xy} R\_demand(\frac{ex-ey}{ed}) = \nabla^2_{yx} R\_demand(\frac{ex-ey}{ed}) = -H^1,$$

where $H^1_{ij} = h_1(ex,ey) = \dfrac{\partial rp\_demand(\frac{ex-ey}{ed})}{\partial(\dfrac{ex-ey}{ed})}\dfrac{1}{ed} \quad \forall i,j \in M$ and $h_1(ex,ey) \le 0$ by using Assumption 3.5.1. Then the Hessian matrix can be formulated as

$$H(x,y) = \begin{pmatrix} H^1 & -H^1 \\ -H^1 & H^1 \end{pmatrix}.$$

Let $z^T = [z_1 \quad z_2]$ where $z_1, z_2 \in \Re^M$. Next we show that H is negative semidefinite; that is, $z^T H(x,y)z \leq 0$ for each $z \in \Re^{2M}$ and $(x,y) \in \Re_+^{M\times M}$:

$$\begin{aligned} z^T H(x,y)z &= [z_1 \quad z_2] \begin{pmatrix} H^1 & -H^1 \\ -H^1 & H^1 \end{pmatrix} [z_1 \quad z_2]^T \\ &= z_1^T H^1 z_1 + z_2^T H^1 z_2 = h_1(ex,ey)[(ez_1)^2 + (ez_2)^2]. \end{aligned}$$

Since $h_1(ex,ey) \leq 0$, $z^T H(x,y)z \leq 0$ for each $z \in \Re^{2M}$. Thus, H is negative semidefinite.∎

Proof of Remark 3.5.7. $(x,y) \in \Re_+^{M\times M}$ where $M := G \times I \times K$. $R\_demand(\frac{ex-ey}{ex})$ is concave in (x,y) iff its Hessian $H := \Re^{2M\times 2M}$ is negative semidefinite. Next we calculate the Hessian of $R\_demand(\frac{ex-ey}{ex})$: Let $H^1, H^2, H^3 := \Re^{M\times M}$, then

$$\nabla^2_{xx} R\_demand(\frac{ex-ey}{ex}) = H^1,$$

$$\nabla^2_{yy} R\_demand(\frac{ex-ey}{ex}) = H^2, \text{ and}$$

$$\nabla^2_{xy} R\_demand(\frac{ex-ey}{ex}) = \nabla^2_{yx} R\_demand(\frac{ex-ey}{ex}) = H^3,$$

where for $(x,y) \in \Re_+^{M\times M}$ and by using Assumption 3.5.1:

(i) $$H^1_{ij} = \frac{\partial rp\_capacity(\frac{ex-ey}{ex})}{\partial(\dfrac{ex-ey}{ex})} \frac{ey^2}{ex^3} \leq 0, \quad \forall i,j \in M.$$

(ii) $$H^2_{ij} = \frac{\partial rp\_capacity(\frac{ex-ey}{ex})}{\partial(\frac{ex-ey}{ex})} \frac{1}{ex} \leq 0, \quad \forall i,j \in M.$$

(iii) $$H^3_{ij} = -\frac{\partial rp\_capacity(\frac{ex-ey}{ex})}{\partial(\frac{ex-ey}{ex})} \frac{ey}{ex^2} \geq 0, \quad \forall i,j \in M.$$

Note that ey^2 denotes the sum of squares for all the elements of a vector y and ex^3 denotes the sum of cubes for all the elements of a vector x in the above equations. Then the Hessian matrix can be formulated as

$$H(x,y) = \begin{pmatrix} H^1 & H^3 \\ H^3 & H^2 \end{pmatrix}.$$

Let $z^T = [z_1 \quad z_2]$ where $z_1, z_2 \in \Re^M$. Next we show that H is negative semidefinite; that is, $z^T H(x,y)z \leq$

0 for each $z \in \Re^{2M}$ and $(x,y) \in \Re_{+}^{M\times M}$:

$$\begin{aligned} z^T H(x,y) z &= [z_1 \quad z_2] \begin{pmatrix} H^1 & H^3 \\ H^3 & H^2 \end{pmatrix} [z_1 \quad z_2]^T \\ &= z_1^T H^1 z_1 + z_2^T H^2 z_2 + z_1^T H^3 z_2 + z_2^T H^3 z_1. \end{aligned}$$

By using $H_{ij}^1 = -\frac{ey}{ex} H_{ij}^3$, $H_{ij}^2 = -\frac{ex}{ey} H_{ij}^3$, and $H_{ij}^3 = h_3(ex, ey) \geq 0$, we have

$$\begin{aligned} z^T H(x,y) z &= -\frac{ey}{ex} z_1^T H^3 z_1 - \frac{ex}{ey} z_2^T H^3 z_2 + z_1^T H^3 z_2 + z_2^T H^3 z_1 \\ &= -\frac{ey}{ex} h_3(x,y)(ez_1)^2 - \frac{ex}{ey} h_3(x,y)(ez_2)^2 + 2h_3(x,y) ez_1 ez_2 \\ &= h_3(x,y)(ez_1 - \frac{ex}{ey} ez_2)(ez_2 - \frac{ey}{ex} ez_1). \end{aligned}$$

In the above equation, one of the three cases hold for ez_1, ez_2:

Case 1 ($ez_1 < \frac{ex}{ey} ez_2$): Then $ez_2 > \frac{ey}{ex} ez_1$ which implies

$$z^T H(x,y) z = h_3(x,y)(ez_1 - \frac{ex}{ey} ez_2)(ez_2 - \frac{ey}{ex} ez_1) < 0.$$

Case 2 ($ez_1 > \frac{ex}{ey} ez_2$): Then $ez_2 < \frac{ey}{ex} ez_1$ which implies

$$z^T H(x,y) z = h_3(x,y)(ez_1 - \frac{ex}{ey} ez_2)(ez_2 - \frac{ey}{ex} ez_1) < 0.$$

Case 3 ($ez_1 = \frac{ex}{ey} ez_2$): Then

$$z^T H(x,y) z = h_3(x,y)(ez_1 - \frac{ex}{ey} ez_2)(ez_2 - \frac{ey}{ex} ez_1) = 0.$$

Hence, $z^T H(x,y) z \leq 0$ for each $z \in \Re^{2M}$ and H is negative semidefinite. ▮

Chapter 4

Strategic Generation Capacity Choice under Demand Uncertainty: Analysis of Nash Equilibria in Electricity Markets

Abstract

We analyze a two-stage game of strategic firms facing uncertain demand and exerting market power in decentralized electricity markets. These firms choose their generation capacities at the first stage while anticipating a perfectly competitive future electricity spot market outcome at the second stage; thus it is a closed loop game. In general, such games can be formulated as an equilibrium problem with equilibrium constraints (EPEC) and examples have been posed in the literature that have multiple or no equilibria. Therefore, it is of interest to define general sets of conditions under which solutions exist and are unique, which would enhance the value of such models for policy and market intelligence purposes. In this chapter, we consider various types of such a closed loop model regarding the underlying price-demand relations (elastic and inelastic demand), the assumed demand uncertainty with a broad class of continuous distributions, and any finite number of players with symmetric or asymmetric costs. We establish sufficient conditions for the random demand's probability distribution which guarantee existence and uniqueness of equilibria in most of the cases of this closed loop model. We identify a broad class of commonly used continuous probability distributions satisfying these conditions.

4.1 Introduction

In this chapter, we establish sufficient conditions which guarantee existence and uniqueness of equilibria in oligopolistic electricity markets where strategic electricity generators (anticipating a perfectly competitive spot market outcome) make capacity choices about future market conditions under demand uncertainty and the power generation is dispatched after the level of demand is realized; thus it is a closed loop game. In general, such games can be formulated as an equilibrium problem with equilibrium constraints (EPEC) which may have multiple or no equilibria. Therefore, it is of interest to define general sets of conditions under which solutions exist and are unique, which would enhance the value of such models for policy and market intelligence purposes (e.g., Schroeder (2012), Allcott (2012)). Furthermore, when firms anticipate a perfectly competitive spot market outcome, an equilibrium of the closed loop[1] model may not be found by solving a less complicated open loop model[1] (see Wogrin et al. (2012)). Thus, it is of interest to users of these models to know whether their assumptions satisfy some conditions which guarantee the existence and uniqueness of an equilibrium before solving such complicated closed loop models.

Short term aspects of market power (related to choices of production when capacities are given) within various specific market design assumptions have been extensively analyzed in the literature. Among these, equilibrium modeling of Nash games in quantities is a common approach (e.g., see Borenstein and Bushnell (1999), Wei and Smeers (1999), Hobbs (2001) for reviews). Long term aspects of market power related to generation capacity expansion have also received growing attention. Among the most important works in the literature, Kreps and Scheinkman (1983) analyze capacity choice at the first stage prior to price competition at the second stage (e.g., Bertrand) with deterministic demand. They show that there exists a unique equilibrium which is equivalent to the single stage Cournot outcomes. Gabszewicz and Poddar (1997) analyze capacity choices of two symmetric firms at the first stage prior to a Cournot market at the second stage with demand uncertainty. They prove existence of symmetric equilibrium and compare it with the deterministic solution of using expected demand. Murphy and Smeers (2005) move one step further and formulate open and closed loop market models with two asymmetric Cournot players and finite number of demand scenarios, each having a linear demand curve with respective probabilities. In their analysis, Murphy and Smeers (2005) conclude that the complexity of solving capacity expansion increases with a closed loop model even without considering the network limitations. They find closed loop examples which may not have an equilibrium. They show that if an equilibrium exists, then it is unique and falls between the perfect competition and open loop equilibria.

All the models mentioned above assume an elastic demand represented by a linear price-demand curve with random intercept and the structure of the assumed probability distribution is in general simple (i.e., uniform demand segments). Furthermore, their analysis are in general based on limited number of players (e.g., two players). In this chapter, we consider both inelastic random demand and an elastic demand represented by a linear price-demand curve with random intercept having a general underlying continuous probability distribution. In addition, we analyze the capacity choices of both symmetric and asymmetric firms. We establish existence and uniqueness results under a broader scope in terms of the underlying price-demand relations (elastic and inelastic demand), the assumed demand uncertainty with a broad class of continuous distributions, and any finite number of players with symmetric or asymmetric

[1]In an open loop model, firms sell electricity production simultaneously with their investment decision, while in a closed loop model firms choose their capacity at the first stage and sell production at the second stage.

costs.

In a similar closed loop model with symmetric firms and elastic demand, Grimm and Zoettl (2008) analyze strategic capacity choices under more general assumptions of a monotone random demand curve and its underlying probability distribution. They establish existence and uniqueness results when firms engage in Cournot competition at the second stage. They also show existence of equilibrium when firms anticipate competitive spot market outcomes at the second stage; however the uniqueness of equilibrium cannot be established under their general assumptions. In our analysis, the closed loop game with symmetric firms facing a linear demand curve with random intercept is indeed a subclass of their model whereas the closed loop model with inelastic random demand constitutes another type. For both cases, we establish sufficient conditions for the random demand's probability distribution which guarantee a unique equilibrium for symmetric firms anticipating competitive spot market outcomes. Furthermore, we establish sufficient conditions for a unique equilibrium of such a closed loop game with asymmetric firms facing an elastic random demand (i.e., a linear demand curve with random intercept). We explain our analysis in more detail below and elaborate on our results and contributions.

In a single stage Cournot model where symmetric firms choose their production output under uncertainty about the intercept of a linear price-demand function, Lagerlöf (2006) shows that if the distribution of the demand has a monotone hazard rate then the uniqueness of equilibrium is guaranteed. In this chapter, we establish an extension of that result for a two-stage game in oligopolistic energy-only electricity markets. As we mention before, we consider strategic firms facing uncertain demand and competing as Cournot players when they choose their generation capacities at the first stage while anticipating a perfectly competitive future spot market outcome based on their choices (for example they may expect regulatory intervention at the spot market). After a firm installs its capacity at the first stage, its production takes place in electricity spot market at the second stage which represents a competitive energy-only market. We look into both symmetric and asymmetric firms facing inelastic or elastic demand. In reality, electricity demand is uncertain and/or fluctuating with a very low elasticity. We focus on two cases of demand uncertainty: (i) an inelastic random demand drawn from a continuous distribution; or (ii) price-demand relationship that can be well approximated by a linear curve with a random intercept having a general underlying continuous probability distribution. Due to demand uncertainty, firms' capacity choices at the first stage may not be utilized for all the demand realizations at the second stage. For such a two-stage game, we characterize the class of problems and continuous probability distributions under which we can show that a unique equilibrium exists. We show that one general class of distributions having a monotone increasing hazard rate are logconcave distributions under which existence and uniqueness of equilibria is guaranteed for symmetric generators facing an inelastic or elastic random demand. When firms have asymmetric costs and the demand curve is elastic (represented by a linear demand curve with random intercept), uniqueness of equilibria can still be established under some sufficient conditions of demand's probability distribution which are similar to a standard assumption used for the demand function itself in the literature (see Sherali et al. (1983), Wolf and Smeers (1997), Grimm and Zoettl (2008), and Xu (2005)).

Finding optimal capacity choices of oligopolistic firms in a closed loop model is in general complex. In the absence of market power, the corresponding two-stage capacity investment problem is a convex problem and, as shown in Chapter 3 of this thesis, most of the two-stage capacity investment equilibrium problems in competitive markets can be cast into convex non(linear) optimization problems similar to the early capacity expansion models. However, convexity is in general lost when market power is introduced. The two-stage capacity investment problem of a strategic firm may be formulated as a bilevel problem, to be more precise as a stochastic MPEC (mathematical program with equilibrium constraints). We consider in this chapter multiple strategic firms having an MPEC problem each, then one can speak of finding a Nash equilibrium to *an equilibrium problem with equilibrium constraints (EPEC)*. It is well known that EPECs are in general nonconvex problems and they may have multiple or no solutions (e.g., Hu (2003), Ehrenmann (2004), Hu and Ralph (2007), Gabriel et al. (2012)). As a consequence, an equilibrium for the corresponding two-stage game may not exist or, if exists, it is in general not unique. Thus, establishing a set of general conditions for existence and uniqueness is desirable for policy and market models including strategic firms so that users can have confidence in their solutions. Throughout the analysis in this chapter, we aim to shed light on the characteristics of the random demand affecting the continuity, differentiability, and generalized concavity of each firm's expected payoff at the first stage which are crucial in establishing the existence and uniqueness of the equilibria of the two-stage strategic capacity choice game.

In the first part of the analysis, we consider an inelastic random demand and strategic firms anticipating a competitive energy-only spot market with VOLL pricing (or price cap). The idea of VOLL pricing is to price electricity at a high value that is supposed to reflect the value of lost load (VOLL) when demand is curtailed. In reality since VOLL is difficult to estimate, a high price cap (which is in general lower than VOLL) is used. As a preliminary, we first establish that whether strategic firms sell their power via central auction or bilateral contracts in the competitive spot market, the corresponding two-stage games yield the same equilibrium, if it exists. A similar result has also been established by Metzler et al. (2003) for a short-run Cournot competition in electricity spot markets where generation capacities are fixed. Metzler et al. (2003) show that a bilateral market model with Cournot producers and perfect arbitrageurs yields the same Nash-Cournot equilibrium (e.g., prices, generation) as a pool market with Cournot players. We extend their result for the two-stage capacity expansion equilibrium problem we consider here. As a result, we continue our analysis with the bilevel problem of each firm under the assumption of a stylized pool market model and in order to preserve analytical tractability we do not consider any network constraints. We note that the continuity of random demand distribution is a necessary condition for the rest of the analysis and the results we present below.

For the bilevel problem of each strategic firm maximizing its expected profit, the constraint set is closed and convex; however, the objective function is in general not concave. Hence, continuity and (strict) quasiconcavity of the objective function are desirable properties for establishing existence and uniqueness results for the corresponding two-stage stochastic game. We first show that each firm's expected profit is continuous provided that the probability distribution of the demand is continuous. Moreover, when firms are symmetric and the distribution of random demand has a monotone hazard rate, each firm's expected profit is strictly quasiconcave and hence the corresponding two-stage game has a unique equilibrium. One class of probability distributions having a monotone increasing hazard functions is logconcave probability distributions. The random demand vector is log-concavely distributed if the logarithm of its probability density function is concave on its support. Logconcave probability distributions constitute a broad class (e.g., the uniform, normal, exponential, logistic, Weibull, gamma). Bagnoli and Bergstrom (2005) give a systematic treatment of the logconcave distributions and their crucial role in a wide variety of economic models. As a result, we show that in case of symmetric firms, random demand with a monotone increasing hazard function is sufficient to guarantee uniqueness of equilibrium. When firms are asymmetric, quasiconcavity of the expected profit function is lost for the ones investing in base-load technologies whereas the objective function of the firm investing in a peak load technology has similar characteristics to that of symmetric firms. Hence, the quasiconcavity of a firm's expected profit is dependent both on the characteristics of the underlying continuous probability distribution of the demand and on whether it is investing in a marginal unit or not. We also show by an example that even with two generators (e.g., peak-load and base-load) facing uniform exogenous demand, the first stage objective function of the base-load generator may not satisfy generalized concavity. Hence, an equilibrium may not exist.

In the second part of our analysis, instead of VOLL pricing we consider a linear price-demand curve with a random intercept, which implies random or fluctuating consumption levels for a given price. Under this setting, we can show that each firm's expected profit is differentiable provided that the probability density function of the intercept is continuous. Moreover when firms are symmetric and the underlying probability density function of the random intercept is logconcave, each firm's expected profit is strictly logconcave. Thus logconcavity of the underlying probability distribution is again a sufficient condition guaranteeing existence and uniqueness of the equilibrium for the two-stage capacity investment game between symmetric firms. When firms are asymmetric, the uniqueness of equilibrium is guaranteed for another condition (different than logconcavity) of probability distribution under which we prove strict concavity of all firms' expected profit functions. The corresponding condition is also similar to a standard assumption in the literature used for the demand function itself (see Sherali et al. (1983), Wolf and Smeers (1997), Grimm and Zoettl (2008), and Xu (2005)). We also conjecture that the logconcavity of probability distributions may also be sufficient to guarantee the existence of a unique equilibrium for closed loop games with asymmetric firms when demand is endogenous. Although we could not establish a theoretical proof for this conjecture, we observed it numerically. In a closed loop game with two asymmetric firms facing an elastic demand, we numerically computed the expected profit function of each firm under various logconcave probability distributions of random intercept (e.g., uniform, normal, exponential, Weibull, gamma, beta) and we observed that each firm's expected profit is strictly quasiconcave in all the cases.

To summarize, we take up a closed loop game in oligopolistic electricity markets which has received growing attention in the literature. In such a game, we formulate the two-stage investment model of multiple strategic firms which make generation capacity choices about future market conditions under demand uncertainty while anticipating a perfectly competitive spot market outcome. We contribute to existing literature along two dimensions:

– We show that whether strategic firms sell their power via central auction or bilateral contracts in the competitive spot market, the corresponding closed loop games yield the same equilibrium, if it exists. Thus, when the spot market is perfectly competitive, different structures of buying and selling electricity do not affect the generation capacity choices of strategic firms.

– In general, such two-stage games can be formulated as an EPEC and examples have been posed in the literature that have multiple or no equilibria. We consider various types of the two-stage investment model of firms regarding the underlying price-demand relations, the assumed demand uncertainty, and any finite number of players with symmetric or asymmetric costs. In most of the cases, we establish sufficient conditions under which solutions exist and are unique and we identify a broad class of commonly used continuous probability distributions of the random demand satisfying these conditions, which will enhance the value of such models for policy and market intelligence purposes. In particular, establishing a set of general conditions for existence and uniqueness is desirable for policy and market models including strategic firms so that users can have confidence in their solutions before solving such complicated closed loop models.

The chapter is organized as follows: In Section 4.2, we introduce the notation and some characteristics of short-run competitive equilibria in spot markets established in the literature. In Section 4.3, we formulate the two-stage game of multiple strategic firms in an energy-only electricity market with VOLL pricing in which we consider two different market structures of buying and selling electricity at the second stage. We first formulate the second stage game representing a spot market with a central auction (pool market) in Section 4.3.1. Then we consider, in Section 4.3.2, a bilateral market where firms trade bilaterally at the second stage. When the spot market is competitive, Boucher and Smeers (2001) show that for given capacities and fixed demand bilateral and pool markets yield the same equilibrium. By using their result, Section 4.3.3 concludes that the two-stage game with pool market and the two stage game with bilateral market yield the same equilibrium (e.g., capacities, prices, generation) when demand is random and capacities are endogenous. We analyze in Section 4.3.4 a simplified pool market model with exogenous random demand. For this simplified model, we show strict quasiconcavity of a firm's expected profit and existence and uniqueness of equilibria under symmetric costs and random demand having logconcave continuous probability distribution. Then in Section 4.4, we replace VOLL pricing with linear price-demand curves with random intercepts. Similar to the exogenous demand case, we show strict log-concavity of a firm's expected profit and existence and uniqueness of equilibria when firms have same costs and the corresponding probability density function is logconcave. When firms have different costs, a sufficient condition which grantees existence and uniqueness of equilibria is also established. We finish the chapter with our conclusions. Appendix 4.A contains the proofs of all theorems, lemmas, and propositions.

4.2 Set up

4.2.1 Notation

We consider an energy-only electricity market model in which there are three types of agents; namely generators, a transmission system operator (TSO), and consumers. We consider this model in a two-stage set-up: firms invest in generation capacity in the first stage and the market clears to satisfy demand in the spot market at the second stage. Firms maximize their profits at the first stage while anticipating a competitive spot market outcome at the second stage. They compete “a la Cournot” while choosing their generation capacities. That is, each firm chooses its generation capacity taking as given the investment strategies of its rivals. In reality, electricity demand is time varying and future demand is uncertain. We represent both phenomena by assuming that demand can take different values in different states of the world $\omega \in \Omega$, each occurring with some probability. The following notation is similar to the one given in Chapter 3 and would apply in state of the world $\omega \in \Omega$:

Sets

N	:	set of all demand nodes
G	:	set of all firms
I_g	:	set of supply nodes of firm $g \in G$
I	:	set of all supply nodes ($I := \cup_g I_g$)
K_g	:	set of plant types of firm $g \in G$
L	:	set of electricity transmission lines in the network

Parameters

c_{ik}^{g}	:	unit generation cost of plant type $k \in K_g$ owned by firm $g \in G$ at supply node $i \in I_g$
κ_k	:	unit capacity cost of plant type $k \in K_g$
$d_n(\omega)$	:	demand at node $n \in N$
$PTDF_{l,j}$	:	power transmitted through line $l \in L$ due to one unit of power injection from node $j \in \{N \cup I\}$ to an arbitrary hub[2] node
h_l	:	capacity limit of line $l \in L$
$VOLL$	:	the value of unserved energy or lost load

[2] $PTDF$ is calculated based on a hub node in $n \in N$ in a standard DC load flow model. The choice of hub node is arbitrary. That is, the flows resulting from a power injection at one node and an equal withdrawal at another do not depend on the location of the hub.

Variables:

Second Stage:

$y_{ik}^g(\omega)$	:	quantity of power generated by plant type $k \in K_g$ of firm $g \in G$ at supply node $i \in I_g$
$f_j(\omega)$	:	net power flow dispatched by TSO from node $j \in \{N \cup I\}$
δ_n	:	unserved (curtailed) energy at node $n \in N$
$p_j(\omega)$	:	locational market price (nodal price) at node $j \in \{N \cup I\}$ which corresponds to shadow price of market clearing constraint

First Stage:

x_{ik}^g	:	the capacity of plant type $k \in K_g$ owned by firm $g \in G$ at node $i \in I_g$.

4.2.2 Characterizations of Perfectly Competitive Spot Markets

The spot market at the second stage may be considered under two different market structures, namely bilateral and pool spot markets. Boucher and Smeers (2001) consider a competitive equilibrium of a game in both pool and bilateral spot markets where all parameters are deterministic and none of the agents (firms, consumers, and TSO) has market power. Boucher and Smeers (2001) also consider, for pool and bilateral spot markets, a corresponding optimization problem referred to as Optimal Power Flow Problem (OPF) which minimizes the total system cost of the spot market. We will use their following result to formulate the second stage problem of each firm and to establish the equivalence of two-stage game in bilateral and pool markets.

Proposition 4.2.1. *[Theorem 1 of Boucher and Smeers (2001)] Consider a power system in a competitive pool or bilateral spot markets when capacities are given and all parameters are deterministic. If there exists an optimal dispatch, then there exists at least one equilibrium, which is a competitive equilibrium, for each of these markets. Moreover:*
(i) *There is one-to-one correspondence between the equilibria of these spot markets; that is any equilibrium of the pool market can be written as an equilibrium of the bilateral spot market and vice versa.*
(ii) *The equilibria of both pool and bilateral spot markets are equivalent to the solution of their associated OPF problems (which are specified later in Sections 4.3.1.1 and 4.3.2.1).*

4.3 Two-stage Capacity Choice Model under Exogenous Random Demand

In this section, we formulate the generation capacity investment model of a strategic firm anticipating the competitive energy-only spot market outcome under demand uncertainty. We focus on the $VOLL$ pricing mechanism in the spot market which gives the firms the incentive to build peak capacity. Whenever there is a shortage of capacity, there is curtailment and the electricity price becomes a very high value (the

$VOLL$ or a price cap) set by the regulator where

$$VOLL > \max\{c_{ik}^{g} | g \in G, i \in I_g, k \in K_g\}.$$

Hence, only when there is curtailment, the peak generator sells power at a price (the $VOLL$) higher than its marginal production cost. The difference between the $VOLL$ and the marginal cost of the peak generator operating contributes to covering its investment cost.

We also consider an exogenous random demand which varies, say, over a year. Let $(\Omega, \mathscr{F}, \Psi)$ denote the common underlying probability space where Ψ represents the joint probability distribution for the random demand vector $d(\omega) := \{d_n(\omega)\}_{n\in N, \omega\in\Omega}$ with $E[|d(\omega)|] < \infty$. Then each $d_n(\omega)$, demand at node $n \in N$, has a general distribution Ψ_n where $\Psi := \prod_{n=1}^{N} \Psi_n$.

Next, we formulate the two-stage game under two different market structures, namely bilateral and pool markets. Each of these market structures gives rise to different types of model formulations as outlined in Sections 4.3.1 and 4.3.2. As mentioned earlier, Boucher and Smeers (2001) address the equivalence of pool and bilateral markets in a single-stage perfectly competitive game when capacities are given and demand is deterministic. Section 4.3.3 is an extension of their result in our setting. First, instead of a single-stage perfectly competitive model we have a two-stage game and the firms may exert market power at the fist stage. Second, the generation capacities are decision variables of the firms at the first stage. Finally, instead of deterministic demand we consider stochastic demand at the second stage. We show in Section 4.3.3 that the two-stage games with pool and bilateral markets, outlined in Sections 4.3.1 and 4.3.2, respectively, yield the same first and the second stage equilibria; hence they are equivalent.

4.3.1 Pool Market Model

We first consider the two-stage game in pool market in which competitive firms sell power to a central auction operated by the Transmission System Operator (TSO) at the price of their supply nodes and the TSO dispatches this power to the consumers at demand nodes. Next, we give details of the two stages in this setting.

4.3.1.1 Perfect Competition Equilibrium at Second Stage

By Proposition 4.2.1 (ii) we know that, in deterministic setting, competitive equilibria of the perfectly competitive pool market is equivalent to the solution of its associated OPF problem. Indeed, this result holds for each state of the world in our setting. For each state of the world $\omega \in \Omega$, one can find a competitive equilibrium of the pool market at the second stage by solving the following OPF (Optimal

Power Flow) problem:

$$Z^*_{pool}(\omega,x) := \min_{\{y(\omega),\delta(\omega),f(\omega)\}} \sum_{g\in G}\sum_{i\in I_g}\sum_{k\in K_g} c^g_{ik} y^g_{ik}(\omega) + VOLL\sum_n \delta_n(\omega)$$

s.t.

$$Cons\_Balance_{pool}(\omega): \boxed{\begin{array}{lll} \sum_{g\in G}\sum_{k\in K_g} y^g_{jk}(\omega) + \delta_j(\omega) + f_j(\omega) \geq d_j(\omega) & (p_j(\omega)) & \forall j \in \{N\cup I\} \\ \\ \sum_{j\in\{N\cup I\}} f_j(\omega) = 0 & (\rho(\omega)) & \end{array}}$$

$$Cons\_Cap(\omega,x): \boxed{x^g_{ik} - y^g_{ik}(\omega) \geq 0 \quad (\beta^g_{ik}(\omega)) \quad \forall g \in G, i \in I_g, k \in K_g}$$

$$Cons\_PTDF(\omega): \boxed{\begin{array}{lll} h_l - \sum_{j\in\{N\cup I\}} PTDF_{l,j} f_j(\omega) \geq 0 & (\lambda^+_l(\omega)) & \forall l \\ h_l + \sum_{j\in\{N\cup I\}} PTDF_{l,j} f_j(\omega) \geq 0 & (\lambda^-_l(\omega)) & \forall l \end{array}}$$

$$y(\omega) \geq 0, \quad \delta(\omega) \geq 0, \tag{4.1}$$

where $\rho(\omega), p_j(\omega), \lambda^+_l(\omega), \lambda^-_l(\omega)$, and $\beta^g_{ik}(\omega)$ are Lagrange multipliers; in particular, $p_j(\omega)$ is the price of unit power (€/MWh) at node $j \in \{N\cup I\}$. In OPF problem (4.1), $Cons\_Balance_{pool}(\omega)$ is the set of energy balance equations which state that the difference between total generation and demand at any node has to be balanced by injections into or withdrawals from the grid and the sum of all injections into and withdrawals from the grid should be zero. $Cons\_PTDF(\omega)$ is the set of constraints for the technical network limits while the generation capacity constraints are given in $Cons\_Cap(\omega)$ which indicate that firms cannot produce more than their maximum capacity. KKT conditions of the OPF problem (4.1), which is a linear program, yield the following equilibrium conditions of the pool market:

For each $\omega \in \Omega$ and x,

$$\begin{array}{llll}
0 \leq c_{ik}^{g} - p_i^*(\omega) + \beta_{ik}^{*g}(\omega) & \perp & y_{ik}^{*g}(\omega) \geq 0 & \forall g \in G, i \in I_g, k \in K_g \\
0 \leq VOLL - p_n^*(\omega) & \perp & \delta_n^*(\omega) \geq 0 & \forall n \in N \\
0 \leq \sum_{g \in G} \sum_{k \in K_g} y_{jk}^{*g}(\omega) + \delta_j^*(\omega) - f_j^*(\omega) - d_j(\omega) & \perp & p_j^*(\omega) \geq 0 & \forall j \in \{N \cup I\} \\
p_j^*(\omega) - \rho^*(\omega) + \sum_{l \in L} PTDF_{l,j}(\lambda_l^{*+}(\omega) - \lambda_l^{*-}(\omega)) = 0 & & & \forall j \in \{N \cup I\} \\
\sum_{j \in \{N \cup I\}} f_j^*(\omega) = 0 & & & \\
0 \leq x_{ik}^{g} - y_{ik}^{*g}(\omega) & \perp & \beta_{ik}^{*g}(\omega) \geq 0 & \forall g \in G, i \in I_g, k \in K_g \\
0 \leq h_l - \sum_{j \in \{N \cup I\}} PTDF_{l,j} f_j^*(\omega) & \perp & \lambda_l^{*+}(\omega) \geq 0 & \forall l \\
0 \leq h_l + \sum_{j \in \{N \cup I\}} PTDF_{l,j} f_j^*(\omega) & \perp & \lambda_l^{*-}(\omega) \geq 0 & \forall l.
\end{array} \tag{4.2}$$

4.3.1.2 First Stage Behavior with Market Power

If firm $g \in G$ operates at node $i \in I_g$, then it sells power to TSO in the spot market at the price of node i. Consequently, it receives a marginal profit equal to unit price $(p_i^*(\omega, .))$ minus unit cost of its production (c_{ik}^{g}) at node i for realization $\omega \in \Omega$. The long run profit of each firm at the first stage is determined by its investment costs. Let x^{-g} be the set of investment decisions of the rival firms. For fixed but arbitrary x^{-g}, the profit of firm $g \in G$ at the first stage for each realization $\omega \in \Omega$ can be formulated as:

$$\Pi_{pool}^{g}(\omega, x^g, x^{-g}) = \sum_{i \in I_g} \sum_{k \in K_g} (p_i^*(\omega, x^g, x^{-g}) - c_{ik}^{g}) y_{ik}^{*g}(\omega, x^g, x^{-g}) - \sum_{i \in I_g} \sum_{k \in K_g} \kappa_k x_{ik}^{g}.$$

For given x^{-g}, firm $g \in G$ determines its optimal investment decision x^{*g} by solving its profit maximization problem:

$$\chi_{pool}^{g}(x^{-g}) := \{x^{*g} | x^{*g} = \underset{x^g \geq 0}{\operatorname{argmax}} \quad E_\omega[\Pi_{pool}^{g}(\omega, x^g, x^{-g})]\}.$$

where $\chi_{pool}^{g}(x^{-g})$ is the set of solutions which maximize firm g's long run profit for given x^{-g}.

4.3.2 Bilateral Market Model

In a bilateral market, competitive firms supply power by bilateral transactions and purchase transmission services for these transactions from TSO who prices transmission capacity based on a congestion pricing scheme. The TSO charges a congestion based wheeling fee ν_j (€/MWh) for transmitting power from an arbitrary hub node to node j. Firm $g \in G$ pays $-v_i$ to get power to the hub from its supply node $i \in I_g$ and v_n to convey power for sale from the hub to customers at node $n \in N$. The additional decision variables of the bilateral market at the second stage are the following:

$s_n^g(\omega)$: quantity of power sold by firm $g \in G$ at demand node $n \in N$ at demand realization $d(\omega)$

$v_j(\omega)$: wheeling fee for transmitting power from an arbitrary hub node to node $j \in \{N \cup I\}$ at demand realization $d(\omega)$.

Next, we formulate the two-stage model of each firm in a bilateral market.

4.3.2.1 Perfect Competition Equilibrium at Second Stage

Again by using Proposition 4.2.1 (ii), we know that the solution of OPF problem (4.3) is a competitive equilibrium of the bilateral spot market for each $\omega \in \Omega$. It can be easily observed that the only difference between OPF problems (4.1) and (4.3) is the energy balance constraints:

$$
\begin{aligned}
Z^*_{bilateral}(\omega,x) :=\quad & \min_{\{s(\omega),y(\omega),\delta(\omega),f(\omega)\}} \sum_{g\in G}\sum_{i\in I_g}\sum_{k\in K_g} c^g_{ik} y^g_{ik}(\omega) + VOLL \sum_{n\in N} \delta_n(\omega) \\
& s.t. \\
Cons\_Balance_{bilateral}(\omega):\quad & \boxed{\begin{array}{lll}
\sum_{g\in G} s^g_n(\omega) + \delta_n(\omega) \geq d_n(\omega) & (p_n(\omega)) & \forall n \in N \\
\sum_{g\in G}\sum_{k\in K_g} y^g_{jk}(\omega) - \sum_{g\in G} s^g_j(\omega) - f_j(\omega) = 0 & (v_j(\omega)) & \forall j \in \{N \cup I\} \\
\sum_{i\in I_g}\sum_{k\in K_g} y^g_{ik}(\omega) - \sum_{n\in N} s^g_n(\omega) = 0 & (\rho_g(\omega)) & \forall g \in G
\end{array}} \\
& y(\omega) \text{ satisfy } Cons\_Cap(\omega,x), \quad f(\omega) \text{ satisfy } Cons\_PTDF(\omega) \\
& y(\omega) \geq 0, \quad \delta(\omega) \geq 0, \quad s(\omega) \geq 0.
\end{aligned}
\tag{4.3}
$$

Again, $\rho_g(\omega), p_n(\omega)$, and $v_j(\omega)$, are Lagrange multipliers; in particular, $p_n(\omega)$ is the price of unit power (€/MWh) at demand node n and, as mentioned earlier, $v_j(\omega)$ is the congestion based wheeling fee (€/MWh) for transmitting power from an arbitrary node to node j. $\rho_g(\omega)$ is the dual variable of the energy balance equation of firm $g \in G$ at the hub; hence its standard interpretation is the marginal cost of firm g at the hub of the linearized DC network. The KKT conditions of the OPF problem (4.3) yield the following equilibrium conditions of the bilateral market:

For each $\omega \in \Omega$ and x,

$$
\begin{array}{lll}
0 \le c_{ik}^{g} - \rho_g^*(\omega) - \nu_i^*(\omega) + \beta_{ik}^{*g}(\omega) & \perp \;\; y_{ik}^{*g}(\omega) \ge 0 & \forall g \in G, i \in I_g, k \in K_g \\
0 \le \rho_g^*(\omega) - p_n^*(\omega) + \nu_n^*(\omega) & \perp \;\; s_n^{*g}(\omega) \ge 0 & \forall g \in G, n \in N \\
0 \le VOLL - p_n^*(\omega) & \perp \;\; \delta_n^*(\omega) \ge 0 & \forall n \in N \\
0 \le \sum_{g \in G} s_n^{*g}(\omega) + \delta_n^*(\omega) - d_n(\omega) & \perp \;\; p_n^*(\omega) \ge 0 & \forall n \in N \\
\nu_j^*(\omega) + \sum_{l \in L} PTDF_{l,j}(\lambda_l^{*+}(\omega) - \lambda_l^{*-}(\omega)) = 0 & & \forall j \in \{N \cup I\} \\
\sum_{i \in I_g} \sum_{k \in K_g} y_{ik}^{*g}(\omega) - \sum_{n \in N} s_n^{*g}(\omega) = 0 & & \forall g \in G \\
\sum_{g \in G} \sum_{k \in K_g} y_{jk}^{*g}(\omega) - \sum_{g \in G} s_j^{*g}(\omega) - f_j^*(\omega) = 0 & & \forall j \in \{N \cup I\} \\
0 \le x_{ik}^{g} - y_{ik}^{*g}(\omega) & \perp \;\; \beta_{ik}^{*g}(\omega) \ge 0 & \forall g \in G, i \in I_g, k \in K_g \\
0 \le h_l - \sum_{j \in \{N \cup I\}} PTDF_{l,j} f_j^*(\omega) & \perp \;\; \lambda_l^{*+}(\omega) \ge 0 & \forall l \\
0 \le h_l + \sum_{j \in \{N \cup I\}} PTDF_{l,j} f_j^*(\omega) & \perp \;\; \lambda_l^{*-}(\omega) \ge 0 & \forall l,
\end{array}
\tag{4.4}
$$

where $\lambda_l^+(\omega)$ and $\lambda_l^-(\omega)$ are Lagrange multipliers of $Cons\_PTDF(\omega)$ and $\beta(\omega)$ is the vector of Langrange multipliers associated to $Cons\_Cap(\omega, x)$.

4.3.2.2 First Stage Behavior with Market Power

At the first stage, each firm maximizes its revenue minus costs. The revenue of a firm from each demand node $n \in N$ is the unit price multiplied by the number of units sold by that firm at that demand node n. There are three components of the firm's total cost; namely investment, production, and shipment costs. The unit shipment cost consists of the wheeling fee for the generation at supply nodes $(-\nu_i)$ and the wheeling fee for the sales at demand nodes (ν_n). For fixed but arbitrary x^{-g}, the profit of firm $g \in G$ in the first stage at each realization $\omega \in \Omega$ is formulated as:

$$
\begin{aligned}
\Pi_{bilateral}^{g}(\omega, x^g, x^{-g}) = \;& \sum_{n \in N} [p_n^*(\omega, x^g, x^{-g}) - \nu_n^*(\omega, x^g, x^{-g})] s_n^{*g}(\omega, x^g, x^{-g}) \\
& - \sum_{i \in I_g} \sum_{k \in K_g} [c_{ik}^{g} - \nu_i^*(\omega, x^g, x^{-g})] y_{ik}^{*g}(\omega, x^g, x^{-g}) - \sum_{i \in I_g} \sum_{k \in K_g} \kappa_k x_{ik}^{g}.
\end{aligned}
$$

Thus for given x^{-g}, each firm $g \in G$ determines its optimal investment decision x^{*g} by solving its profit maximization problem:

$$
\chi_{bilateral}^{g}(x^{-g}) := \{x^{*g} | x^{*g} = \underset{x^g \ge 0}{\operatorname{argmax}} \quad E_\omega[\Pi_{bilateral}^{g}(\omega, x^g, x^{-g})]\},
$$

where $\chi^g_{bilateral}(x^{-g})$ is the set of solutions which maximize firm g's long run profit for given x^{-g}.

4.3.3 Equivalence of Two-stage Bilateral and Pool Market Models

In this section, we show that the capacity choice set of the firms anticipating the competitive outcome of a pool market is equivalent to that of the firms anticipating the competitive outcome of a bilateral market. Proposition 4.2.1 (i) addresses the equivalence of the equilibria in bilateral and pool markets (i.e., same prices, production quantities, and scarcity rents) when capacities are given. By using this result in our setting, we can easily conclude that for given any x and $\omega \in \Omega$:

- $p^{*p}(\omega, x^g, x^{-g}) = p^{*b}(\omega, x^g, x^{-g})$,

- $y^{*pg}(\omega, x^g, x^{-g}) = y^{*bg}(\omega, x^g, x^{-g})$,

- $\beta^{*pg}(\omega, x^g, x^{-g}) = \beta^{*bg}(\omega, x^g, x^{-g})$,

where we put superscripts p and b to indicate the variables in pool and bilateral markets, respectively. By using the above equalities, we next show that when demand is stochastic and generation capacities are endogenously chosen at the first stage in which firms may exert market power, the first and the second stage problems still yield the same equilibria (i.e., same capacities, prices, production quantities, and scarcity rents) for both market models.

Proposition 4.3.1. *For a firm anticipating the competitive spot market outcome of the the second stage, $E_\omega[\Pi^g_{pool}(\omega, x^g, x^{-g})] = E_\omega[\Pi^g_{bilateral}(\omega, x^g, x^{-g})]$ holds for all $g \in G$ at any given (x^g, x^{-g}). This implies that the two-stage game models with pool and bilateral markets not only yield the same second-stage equilibria at each ω but also the solution sets of the first stage game in these markets are equal, which implies both markets yield the same first stage equilibria.*

Proof. See Appendix 4.A.

Proposition 4.3.1 also holds when firms are Cournot players at the second stage by using the equivalence result of Metzler et al. (2003) (i.e., Theorem 1) for bilateral and pool spot market models. When capacities are given, Metzler et al. (2003) show that Cournot generators in bilateral spot markets with perfectly competitive arbitrageurs yield the same Nash equilibrium as Cournot competition in pool spot markets. Due to the equivalence of pool and bilateral market models, we continue our analysis with pool market model. Thus, we ignore the superscript p in the corresponding variables and use $\Pi^g(\cdot)$ to refer to the profit function of firm $g \in G$ in pool market model.

4.3.4 Characterization of the Two-stage Game

In this section, we investigate the characteristics of the expected profit function of a strategic firm anticipating a competitive market outcome under a continuous random demand with a general distribution. In order to preserve analytical tractability, we consider an electricity market without any network limitations; this in turn collapses the problem to a single node case. In such a single node market, there are several suppliers coping with an aggregated demand. Let $D = \sum\limits_{n \in N} d_n$ denote the aggregated continuous random demand. The following assumptions hold for the two-stage model.

Assumption 4.3.2. *D is a continuous random demand with a cumulative distribution Ψ and a continuous probability density function Ψ' whose support is on some interval $[\underline{D}, \overline{D}]$.*

Assumption 4.3.3. *(i) There is a single firm at each supply node and each firm invests in only one technology; that is $\mid G \mid = \mid I \mid = \mid K \mid$. Firm 1 owns the cheapest generator and firm K owns the peak generator where $c_1 < c_2 < \ldots < c_K$ and $\kappa_1 > \kappa_2 > \ldots > \kappa_K$.*
(ii) For at least one technology $k \in K$, it holds that $VOLL > c_k + \kappa_k$.

From now on, we use $x_{-k} := (x_j)_{j=1, j \neq k}^K$ to denote the vector of strategies of the rival firms of firm $k \in K$ and $x_{-k}^* := (x_j^*)_{j=1, j \neq k}^K$ is the vector of their optimal decisions at equilibrium.

The continuity of the probability density function in Assumption 4.3.2 is crucial for establishing our results in this section. Furthermore, Assumption 4.3.3 (ii) guarantees that the equilibrium of the two-stage game is nontrivial. Let the total generation capacity be lower than the minimum demand ($\sum\limits_{j=1}^K x_j < \underline{D}$). Then, the market prices would be equal to $VOLL$ for all demand realizations in the perfectly competitive spot market. According to Assumption 4.3.3 (ii), $VOLL$ is higher than the unit investment and operation cost for at least one technology, say $k' \in K$. This, in turn, implies that when there isn't sufficient capacity for any demand realization, the expected scarcity rent of at least one firm investing in technology k', which is equal to $VOLL - c_{k'}$, is higher than its unit investment cost, $\kappa_{k'}$. Therefore, for given $x_{-k'} < \underline{D}$, $\frac{\partial E_\omega[\Pi^{k'}(\omega, x_{k'}, x_{-k'})]}{\partial x_{k'}} = VOLL - c_{k'} - \kappa_{k'} > 0$ for $x_{k'} \in [0, \underline{D} - x_{-k'})$. Thus, firm k' would be able to increase its expected profit by investing in technology k' until there is sufficient capacity at least for the minimum demand level. Thus, Assumption 4.3.3 (ii) guarantees that total investment capacity at equilibrium satisfies at least the minimum demand: $\sum\limits_{j=1}^K x_j^* \geq \underline{D}$.

As given in Section 4.3.1.2, for each fixed and arbitrary x_{-k}, firm $k \in K$ obtains its expected profit

$E_\omega[\Pi^k(\omega, x_k, x_{-k})]$ and determines its optimal strategy by solving its profit maximization problem:

$$\max_{x_k \geq 0} \quad E_\omega[\Pi^k(\omega, x_k, x_{-k})]. \tag{4.5}$$

For any given x_{-k}, let $\chi^k(x_{-k})$ denote the solution set of problem (4.5). Then a Nash equilibrium is a point such that $x_k^* \in \chi^k(x_{-k}^*)$ for all $k \in K$. Next, we assess the properties of $E_\omega[\Pi^k(\omega, x_k, x_{-k})]$ in (4.5). Based on the corresponding properties, we will analyze the existence and uniqueness of equilibria $x_k^* \in \chi^k(x^*{}_k)$ for all $k \in K$.

From the proof of Proposition 4.3.1 given in Appendix 4.A, one can easily see that, at the first stage, the expected profit function of each firm $k \in K$ for given x_{-k} can be formulated in terms of its expected scarcity rent:

$$E_\omega[\Pi^k(\omega, x_k, x_{-k})] = (E_\omega[\beta_k^*(\omega, x_k, x_{-k})] - \kappa_k)x_k.$$

For given (x_k, x_{-k}) at the second stage, Figure 4.1 illustrates the stepwise supply curve and some demand realizations within $[\underline{D}, \overline{D}]$ in a single-node spot market. As illustrated in Figure 4.1 unless the capacity of all firms are fully utilized at realization $\omega \in \Omega$, the market price $(p^*(\omega, x_k, x_{-k}))$ is equal to one of the marginal generating cost values where the corresponding demand realization crosses the stepwise supply curve. Otherwise, the market price is equal to $VOLL$. Note that when demand is elastic with a random inverse demand curve, the market prices can also take values between c_k and c_{k+1} for all $k \in \{1, \ldots, K-1\}$ or between c_K and $VOLL$ for some realizations $\omega \in \Omega$ as illustrated in Section 4.4.2 (See Figures 4.3 and 4.4).

Next, we derive a closed form expression for the expected scarcity rent of firm k and establish two basic functional properties; namely its continuity and monotonicity in x_k. For given x_{-k}, we denote the total capacity of the $(m-1)$ cheapest firms excluding firm k in the market as $X_{m-1}(k) = \sum_{j=1, j \neq k}^{m} x_j$, for $k \leq m \leq K$. Then $X_{m-1}(k) + x_k$ is the total capacity of the m cheapest firms including firm k in the market.

Proposition 4.3.4. *Under Assumption 4.3.2, for given x_{-k}, the expected scarcity rent of firm $k \in K$ can be formulated in a closed form expression as follows:*

$$E_\omega[\beta_k^*(\omega, x_k, x_{-k})] = VOLL - c_k - (VOLL - c_K)\Psi(x_k + X_{K-1}(k)) - \sum_{m=k}^{K-1}(c_{m+1} - c_m)\Psi(x_k + X_{m-1}(k)). \tag{4.6}$$

Proof. See Appendix 4.A.

We may also write expression (4.6) in terms of probabilities (that are both driven from equation (4.20)

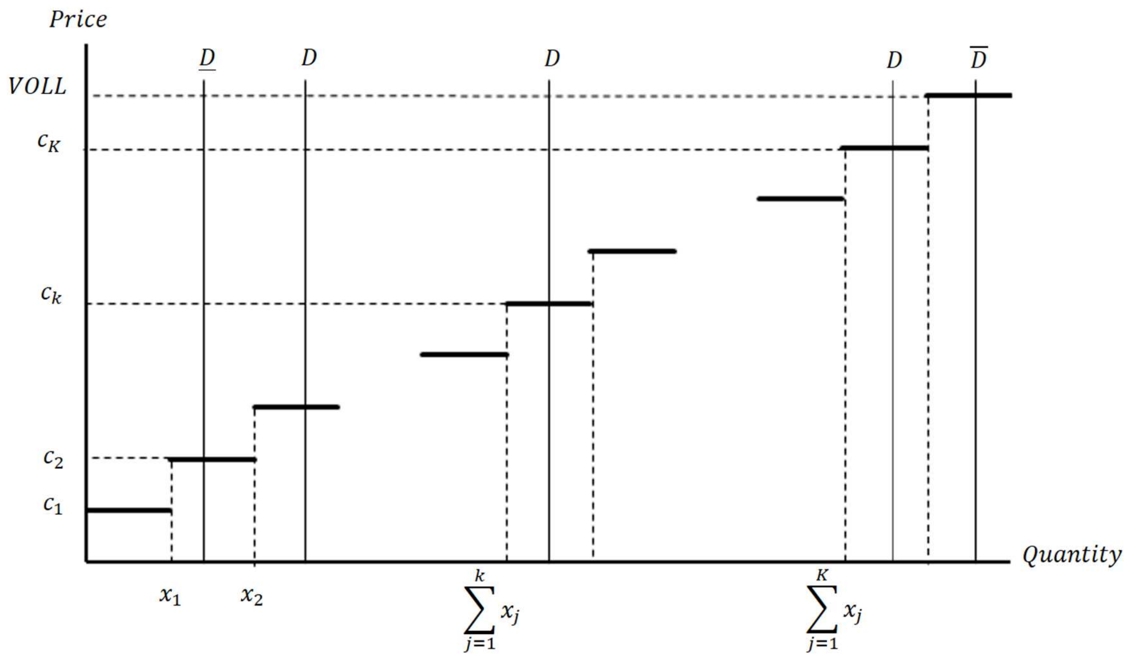


Figure 4.1: Illustration of a single-node market with stepwise supply function for given x and exogenous random demand

given in *Proof of Proposition 4.3.4* in Appendix 4.A) which makes it easier to interpret:

$$\begin{aligned} E_\omega[\beta_k^*(\omega, x_k, x_{-k})] &= \sum_{m=k}^{K-1}(c_{m+1}-c_k)P(X_{m-1}(k)+x_k \le D \le X_m(k)+x_k) \\ &\quad +(VOLL-c_k)P(D \ge X_{K-1}(k)+x_k). \end{aligned}$$

The above expression indicates the following: Firm k receives a positive scarcity rent either when there is at least one other firm operating with a higher marginal generation cost in the market or when demand exceeds the total generation capacity and the market price is set at $VOLL$. The price of the electricity is c_{m+1} with probability $P(X_{m-1}(k)+x_k \le D \le X_m(k)+x_k)$ in which case the firm operating with the highest marginal generation cost is using the technology $m+1$, $m \ge k$. The price of the electricity may go up to $VOLL$ with probability $P(D \ge X_{K-1}(k)+x_k)$ in which case the demand is so high that there is not enough generating capacity in the market. Since the marginal generation cost of firm k is c_k, its scarcity rent from the second stage would be $(c_{m+1}-c_k)$ with probability $P(X_{m-1}(k)+x_k \le D \le X_m(k)+x_k)$, $m \ge k$, and $(VOLL-c_k)$ with probability $P(D \ge X_{K-1}(k)+x_k)$. Note that for given x_{-k}, the values of these probabilities depend on x_k which changes the corresponding ranges.

Lemma 4.3.5. *Under Assumption 4.3.2, the following properties hold for the expected scarcity rent func-*

tion (4.6) of firm $k \in K$:

(i) $E_{\omega}[\beta_k^*(\omega, x_k, x_{-k})]$ *is a continuous function of* $(x_k, x_{-k}) \in \mathscr{R}_+^{\mathscr{K}}$*, and*

(ii) $E_{\omega}[\beta_k^*(\omega, x_k, x_{-k})]$ *is nonincreasing in* $x_k \in \mathscr{R}_+$.

Proof. See Appendix 4.A.

Next, by using the functional properties of the expected scarcity rent function of firm $k \in K$, we will explore the functional properties - such as continuity, differentiability and concavity - of the expected profit function of firm k. In doing so, we will differentiate between two cases. The first one is the simpler case of symmetric firms using the same technology; hence having the same operational and investment costs. We show that when firms are symmetric, the expected profit of each firm is continuous and strictly quasiconcave under certain conditions and there exists a unique Nash equilibrium. Then we will consider asymmetric firms whose expected profit functions are more complicated and may be analytically intractable in general.

4.3.4.1 Symmetric Firms

In the symmetric case, the unit production and investment costs of all firms have the same value; that is, $c_1 = c_2 = \ldots = c_K = c$ and $\kappa_1 = \kappa_2 = \cdots = \kappa_K = \kappa$. This yields an identical profit function for all firms, $E_{\omega}[\Pi^k(\omega, x_k, x_{-k})] = (E_{\omega}[\beta_k^*(\omega, x_k, x_{-k})] - \kappa)x_k$, and (4.6) reduces to (4.7):

$$E_{\omega}[\beta_k^*(\omega, x_k, x_{-k})] = (VOLL - c)(1 - \Psi(x_k + X_{K-1}(k)), \tag{4.7}$$

where $X_{K-1}(k) = \sum_{j=1, j \neq k}^{K} x_j$ is the total capacity of all the rival firms. Next, we show in Lemma 4.3.6 that the expected scarcity rent function of each firm k in (4.7) is differentiable with respect to x_k almost everywhere except at three breakpoints $x_k \in \{0, \max(0, \underline{D} - X_{K-1}(k)), \max(0, \overline{D} - X_{K-1}(k))\}$. Among these breakpoints, $x_k^{min} = \underline{D} - X_{K-1}(k)$ is the investment level below which the probability of demand exceeding total investment capacity is 1 and $x_k^{max} = \overline{D} - X_{K-1}(k))$ is the investment level above which probability of demand exceeding total investment capacity is zero.

Lemma 4.3.6. *For given* x_{-k}*, if* Ψ *is differentiable on* $[\underline{D}, \overline{D}]$*, then* $E_{\omega}[\beta_k^*(\omega, x_k, x_{-k})]$ *in (4.7) is differentiable w.r.t.* $x_k \in \mathscr{R}_+$ *almost everywhere except* $x_k \in \{0, x_k^{min}, x_k^{max}\}$.

Proof. See Appendix 4.A.

In Theorems 4.3.8 and 4.3.10 we show that $E_\omega[\Pi^k(\omega, x_k, x_{-k})]$ is a continuous function and (strictly) quasiconcave in x_k under certain conditions on the cumulative distribution function Ψ. It is well known that games possess a Nash equilibrium if (1) the strategy spaces are nonempty, convex and compact, and (2) players have continuous and quasiconcave payoff functions [e.g., Debreu (1952), Fudenberg and Tirole (1991)]. For completeness, we give the result by Debreu (1952) in the following proposition. This result can be used to establish existence of a Nash equilibrium for our symmetric game which is given by Theorem 4.3.13.

Proposition 4.3.7. [Debreu (1952)] *An n-persons strategic game has a Nash equilibrium if the strategy spaces $S_i, i = 1 \ldots n$, in this game are non-empty, compact, convex subsets of a Euclidian space and the payoff functions u_i are continuous at s and quasiconcave in s_i where $s_i \in S_i$ is the strategy of player i.*

Theorem 4.3.8. *Let Ψ be twice continuously differentiable function on $[\underline{D}, \overline{D}]$ where $\Psi^{'}$ and $\Psi^{''}$ are the first and second derivatives of Ψ. Then the expected profit function of each firm $k \in K$ has the following properties:*

(i) $E_\omega[\Pi^k(\omega, x_k, x_{-k})]]$ is continuous at $(x_k, x_{-k}) \in \mathscr{R}_+^{\mathscr{K}}$;

(ii) For given x_{-k}, $E_\omega[\Pi^k(\omega, x_k, x_{-k})]$ is differentiable w.r.t. $x_k \in \mathscr{R}_+$ almost everywhere except $x_k \in \{0, x_k^{min}, x_k^{max}\}$;

(iii) For given x_{-k}, $E_\omega[\Pi^k(\omega, x_k, x_{-k})]$ is (strictly) concave for $x_k \in (x_k^{min}, x_k^{max})$ iff $-\Psi^{''}(s+C)s - \Psi^{'}(s+C)\ (<) \leq 0$ for $\forall s, C \in \mathscr{R}_+$ where C is some constant. Whenever this condition holds, $E_\omega[\Pi^k(\omega, x_k, x_{-k})]$ is a (strictly) quasiconcave function of x_k on $\mathscr{R}_+$.

Proof. See Appendix 4.A.

Remark 4.3.9. If the probability density function, $\Psi^{'}$, is nondecreasing, then $-\Psi^{''}(x_k + X_{K-1})x_k - \Psi^{'}(x_k + X_{K-1}) \leq 0$ holds for all $x_k \in \mathscr{R}_+$. Therefore, the nondecreasing property of a probability density function is a sufficient condition for quasiconcavity of $E_\omega[\Pi^k(\omega, x_k, x_{-k})]$ on $[0\ \infty)$. In particular, the uniform distribution satisfies this property.

The condition stated in *(iii)* of Theorem 4.3.8 is similar to a standard assumption used in the literature (see Sherali et al. (1983), Wolf and Smeers (1997), Grimm and Zoettl (2008), and Xu (2005)). In these references, the corresponding assumption is given for a random inverse demand function. In our setting with random exogenous demand, Theorem 4.3.8 indicates that a similar condition needs to be assumed for the cumulative probability function of demand. Aside from this standard condition, we identify, in the next theorem, a large class of probability distributions having a monotone increasing hazard function that guarantees the strict quasiconcavity of $E_\omega[\Pi^k(\omega, x_k, x_{-k})]$ w.r.t. x_k. In particular, logconcave probability distributions have a monotone increasing hazard function which is a milder condition than the condition stated in Remark 4.3.9. We will come back to this in Remark 4.3.14.

Theorem 4.3.10. *Under Assumption 4.3.2, let the hazard function of demand, $H(s) = \Psi'(s)/(1-\Psi(s))$, be monotone increasing on $[\underline{D}, \overline{D}]$. Then for any $k \in K$ and x_{-k}, $E_\omega[\Pi^k(\omega, x_k, x_{-k})]$ is strictly quasiconcave for $x_k \in [x_k^{min}, x_k^{max}]$ which implies that it is also strictly quasiconcave w.r.t. x_k on $\mathscr{R}_+$.*

Proof. See Appendix 4.A.

The strategy space of each firm $k \in K$ can be represented by a nonempty, compact, and convex set which is shown in the next lemma. Then in Lemma 4.3.12, we establish the boundaries for the total generation capacity at the first-stage equilibrium problem. Finally, in Theorem 4.3.13, we establish the existence and uniqueness result and characterize the Nash equilibrium for the generation capacity investment game of the symmetric firms.

Lemma 4.3.11. *The strategy space of each firm $k \in K$ is $S_k := [0, \overline{D}]$ which is nonempty, compact, and convex.*

Proof. See Appendix 4.A.

Lemma 4.3.12. *Assume that there exists an equilibrium, $(x_k^*)_{k=1}^K$, for the symmetric game. Then, under Assumption 4.3.3, it holds that $\underline{D} \leq \sum_{j=k}^{K} x_k^* \leq \overline{D}$.*

Proof. See Appendix 4.A.

Theorem 4.3.13 (Existence and Uniqueness). *Under Assumptions 4.3.3 and 4.3.2, let the hazard function of demand, $H(s) = \Psi'(s)/(1-\Psi(s))$, be monotone increasing on $[\underline{D}, \overline{D}]$. Then for the games defined in Sections 4.3.1.2 and 4.3.2.2 with symmetric firms and unlimited transmission capacity, there exists a unique symmetric Nash equilibrium, $x^* = x_1^* = \ldots = x_K^*$, which satisfies*

$$x^* = \frac{(VOLL - c)(1 - \Psi(Kx^*)) - \kappa}{(VOLL - c)\Psi'(Kx^*)} \tag{4.8}$$

and $\underline{D} \leq Kx^* \leq \overline{D}$.

Proof. See Appendix 4.A.

Remark 4.3.14. By Corollary 2 of Bagnoli and Bergstrom (2005), we know that if the probability density function $\Psi'(s)$ is logconcave on $[\underline{D}, \overline{D}]$, then the hazard function $H(s)$ is monotone increasing on $[\underline{D}, \overline{D}]$. Hence, the log-concavity of probability density function of demand implies strict quasiconcavity of $E_\omega[\Pi^k(\omega, x_k, x_{-k})]$ w.r.t. x_k for each k. By Remark 1 of Bagnoli and Bergstrom (2005), the converse is not true because there exists probability distributions with monotone increasing hazard functions but without logconcave density functions.

By Remark 4.3.14, we can identify a broad class of commonly used continuous probability distributions which guarantee the strict quasiconcavity of profit function of each firm at the first stage and hence the existence and uniqueness of Nash equilibria characterized as in Theorem 4.3.13. Bagnoli and Bergstrom (2005) list a number of distributions with logconcave density functions in their Table 1. These include several widely used distributions such as uniform, normal, exponential, Gamma (parameter ≥ 1), Weibull (parameter ≥ 1), Beta (both parameters ≥ 1), among others.

4.3.4.2 Asymmetric Firms

In the asymmetric case, firms may not have identical profit functions since each firm's marginal generation and investment costs are different. In the symmetric case, we have seen that the expected scarcity rent function of each firm is a continuous piecewise function with at most three breakpoints including zero. In the asymmetric case, for each firm $k \in K$, $E_\omega[\beta_k^*(\omega, x_k, x_{-k})]$ will be a piecewise continuous function which may have between 3 and $2(K-1)+3$ breakpoints. The number of breakpoints depends on the position of the firm in the merit order in the spot market. A firm with the highest marginal generation cost (c_K) will have a similar expected scarcity rent function with at most 3 breakpoints as in the symmetric case whereas a firm with the lowest marginal generation cost (c_1) will have at most $2(K-1)+3$ breakpoints. To generalize, a firm $k \in K$ with marginal generation cost c_k will have at most $2(K-k)+3$ breakpoints which can be calculated by doing a careful accounting of the breakpoints in (4.6). Let $\underline{x_k}(m) = \max(0, \underline{D} - X_m(k))$ and $\overline{x_k}(m) = \max(0, \overline{D} - X_m(k))$. The breakpoints of the firm k's expected scarcity rent function will be $0, \{\underline{x_k}(m)\}_{m=k-1}^{K-1}, \{\overline{x_k}(m)\}_{m=k-1}^{K-1}$.

As a result of expected scarcity rent being a piecewise continuous function, $E_\omega[\Pi^k(\omega, x_k, x_{-k})]$ is also piecewise and continuous for each firm k. Moreover, $E_\omega[\Pi^k(\omega, x_k, x_{-k})]$ consists of at most $2(K-k)+4$ function pieces with $2(K-k)+3$ breakpoints. To establish conditions that guarantee quasiconcavity of $E_\omega[\Pi^k(\omega, x_k, x_{-k})]$ w.r.t. x_k for each firm k under a general distribution is more complicated here. This can be done for $k = K$,

$$E_\omega[\beta_K^*(\omega, x_K, x_{-K})] = (VOLL - c_K)(1 - \Psi(x_K + X_{K-1}(k))$$

which is similar to (4.7). Therefore, the results in Theorems 4.3.8, 4.3.10, 4.3.13 and Remarks 4.3.9 and

4.3.14 also hold for the profit function of the most expensive firm ($k = K$) but not necessarily for the other firms ($k < K$).

Next, we give an example with two firms (a base-load and a peak generator) to illustrate the expected scarcity rent and expected profit functions of these firms when demand is uniformly distributed. The example clearly illustrates that for when the demand is uniformly distributed, the expected scarcity rent functions of the firms are piecewise linear. However even in this very simple case, the expected profit function of the base-load generator is not necessarily quasiconcave.

Assume that there are two generators: base-load ($g = 1$) and peak ($g = 2$) with $c_1 < c_2$ and $\kappa_1 > \kappa_2$. Let the transmission capacities be infinite ($h = \infty$) and the aggregated demand of all nodes be uniformly distributed on $[d_{min}, d_{max}]$; that is $d \sim U[d_{min}, d_{max}]$. Next, we give the formulation of the expected scarcity rent function of each firm given a investment strategy of its rival firm. Without loss of generality, we will give the formulations when the given investment strategies satisfy $x_1 < d_{min}$ and $x_2 < d_{min}$ since this will yield the maximum number of breakpoints for the expected scarcity rent and the expected profit functions for both firms. Depending on the values of x_1 and x_2, the number of breakpoints may be lower but the formulations of corresponding function segments will remain the same.

Lemma 4.3.15. *For given $x_2 \geq 0$, there are constants $a^1_{\beta_1}, a^2_{\beta_1}, a^3_{\beta_1} \in R$ and $b^1_{\beta_1}, b^2_{\beta_1}, b^3_{\beta_1} > 0$ which satisfy*

$$E_\omega[\beta_1^*(\omega, x_1, x_2)] = \begin{cases} VOLL - c_1, & \text{if} \quad 0 \leq x_1 < d_{min} - x_2 \\ a^1_{\beta_1} - b^1_{\beta_1} x_1, & \text{if} \quad d_{min} - x_2 \leq x_1 < d_{min} \\ a^2_{\beta_1} - b^2_{\beta_1} x_1, & \text{if} \quad d_{min} \leq x_1 < d_{max} - x_2 \\ a^3_{\beta_1} - b^3_{\beta_1} x_1, & \text{if} \quad d_{max} - x_2 \leq x_1 < d_{max} \\ 0 & \quad\quad \text{otherwise}. \end{cases}$$

Similarly for given $x_1 \geq 0$, there are constants $a_{\beta_2} \in R$, and $b_{\beta_2} > 0$ satisfying

$$E_\omega[\beta_2^*(\omega, x_2, x_1)] = \begin{cases} VOLL - c_2, & \text{if} \quad 0 \leq x_2 < d_{min} - x_1 \\ a_{\beta_2} - b_{\beta_2} x_2, & \text{if} \quad d_{min} - x_1 \leq x_2 < d_{max} - x_1 \\ 0 & \quad\quad \text{otherwise}. \end{cases}$$

Furthermore, these constants can be explicitly found.

Proof. See Appendix 4.A.

Obviously, by using Lemma 4.3.15, the expected profit functions of firm 1 and 2 can also be explicitly formulated as follows (see the proof of Lemma 4.3.15 in Appendix 4.A for the associated constants):
For given $x_2 \geq 0$,

$$E_\omega[\Pi_1(\omega,x_1,x_2)] = \begin{cases} (VOLL - c_1 - \kappa_1)x_1, & \text{if} \quad 0 \leq x_1 < d_{min} - x_2 \\ (a^1_{\beta_1} - \kappa_1)x_1 - b^1_{\beta_1}x_1^2, & \text{if} \quad d_{min} - x_2 \leq x_1 < d_{min} \\ (a^2_{\beta_1} - \kappa_1)x_1 - b^2_{\beta_1}x_1^2, & \text{if} \quad d_{min} \leq x_1 < d_{max} - x_2 \\ (a^3_{\beta_1} - \kappa_1)x_1 - b^3_{\beta_1}x_1^2, & \text{if} \quad d_{max} - x_2 \leq x_1 < d_{max} \\ -\kappa_1 x_1 & \text{otherwise.} \end{cases}$$

Similarly for given $x_1 \geq 0$,

$$E_\omega[\Pi_2(\omega,x_2,x_1)] = \begin{cases} (VOLL - c_2 - \kappa_2)x_2, & \text{if} \quad 0 \leq x_2 < d_{min} - x_1 \\ (a_{\beta_2} - \kappa_2)x_2 - b_{\beta_2}x_2^2, & \text{if} \quad d_{min} - x_1 \leq x_2 < d_{max} - x_1 \\ -\kappa_2 x_2 & \text{otherwise.} \end{cases}$$

Lemma 4.3.16. *$E_\omega[\Pi_1(\omega,x_1,x_2)]$ and $E_\omega[\Pi_2(\omega,x_2,x_1)]$ are continuous functions of $x_1 \in \mathscr{R}_+$ and $x_2 \in \mathscr{R}_+$, respectively; however $E_\omega[\Pi_1(\omega,x_1,x_2)]$ is not differentiable at points $x_1 \in \{d_{min} - x_2, d_{min}, d_{max} - x_2, d_{max}\}$ and $E_\omega[\Pi_2(\omega,x_2,x_1)]$ is not differentiable at points $x_2 \in \{d_{min} - x_1, d_{max} - x_1\}$. Moreover, $E_\omega[\Pi_2(\omega,x_2,x_1)]$ is quasiconcave in x_2.*

Proof. See Appendix 4.A.

Remark 4.3.17. For the above example, each function piece of $E_\omega[\Pi_1(\omega,x_1,x_2)]$ is concave. Let $\frac{\partial^+ E_\omega[\Pi_1(\omega,x_1,x_2)]}{\partial x_1}$ and $\frac{\partial^- E_\omega[\Pi_1(\omega,x_1,x_2)]}{\partial x_1}$ be the right-hand and left-hand side derivatives w.r.t. x_1, respectively. Then, $E_\omega[\Pi_1(\omega,x_1,x_2)]$ is quasiconcave in x_1 if the following condition holds at each breakpoint $x_1 \in \{d_{min} - x_2, d_{min}, d_{max} - x_2, d_{max}\}$:

$$\text{If } \frac{\partial^- E_\omega[\Pi_1(\omega,x_1,x_2)]}{\partial x_1} < 0 \text{ then } \frac{\partial^+ E_\omega[\Pi_1(\omega,x_1,x_2)]}{\partial x_1} < 0, \tag{4.9}$$

which implies whenever $E_\omega[\Pi_1(\omega,x_1,x_2)]$ starts to decrease, it does not increase again.

Since we know the constants explicitly, we can calculate the right and left derivatives at each breakpoint and identify the cases when $E_\omega[\Pi_1(\omega, x_1, x_2)]$ will be quasiconcave. For the above example, all the breakpoints, except $x_1 = d_{max} - x_2$, always satisfy condition (4.9) regardless of the constant values. Thus for $x_1 = d_{max} - x_2$, the constants $a^2_{\beta_1}, a^3_{\beta_1}, b^2_{\beta_1}, b^3_{\beta_1}$ satisfying condition (4.9) guarantee the quasiconcavity of $E_\omega[\Pi_1(\omega, x_1, x_2)]$:

$$\begin{aligned} &\text{If } \frac{\partial^- E_\omega[\Pi_1(\omega, d_{max} - x_2, x_2)]}{\partial x_1} = (a^2_{\beta_1} - \kappa_1) - 2b^2_{\beta_1}(d_{max} - x_2) < 0, \\ &\text{then } \frac{\partial^+ E_\omega[\Pi_1(\omega, d_{max} - x_2, x_2)]}{\partial x_1} = (a^3_{\beta_1} - \kappa_1) - 2b^3_{\beta_1}(d_{max} - x_2) < 0. \end{aligned} \tag{4.10}$$

Since there are two firms in the example above, the expected profit functions of base-load and peak generators are piecewise continuous functions with maximum 5 and 3 breakpoints, respectively. Since the demand is uniformly distributed, each function piece is linear or quadratic. However, even for uniform demand the expected profit function of base-load generator may not be quasiconcave if it does not satisfy the condition given in (4.10). For instance, let $[d_{min}, d_{max}] := [100, 500]$ and $VOLL = 10000, c_1 = 10, c_2 = 200, \kappa_1 = 20$, then when $x_2 = 400$, condition (4.10) does not hold; hence $E_\omega[\Pi_1(\omega, x_1, x_2)]$ is not quasiconcave as shown in Figure 4.2. Thus, the conditions of Theorem 4.3.7 for existence of equilibrium are not satisfied.

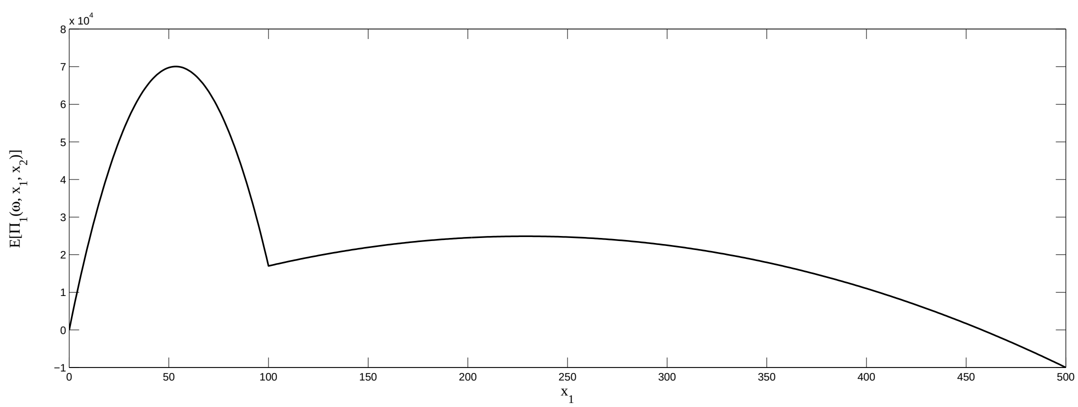


Figure 4.2: Expected profit function of firm 1 (base-load generator) at $x_2 = 400$

4.4 Two-stage Capacity Choice Model with Endogenous Random Linear Price-Demand Curve

In this section, instead of VOLL pricing we consider an electricity market where consumers can respond to prices and give firms an incentive to build generation capacity. When consumers respond to prices, we have an elastic demand and we represent the reaction of consumers to the prices by decreasing linear price-demand curves. The standard linear inverse demand function with random parameters may be represented by $P_n(\omega, d_n)$:

$$P_n(\omega, d_n) = \alpha_n(\omega) - \gamma_n(\omega) d_n,$$

where $P_n(\omega, 0) = \alpha_n(\omega) < \infty, \forall \omega$. For given ω, d_n is not fixed as in Section 4.3 but it is a decision variable which depends on the price of electricity at node n.

We focus on the case of random intercept: $P_n(\omega, d) = \alpha_n(\omega) - \gamma_n d_n$. This a standard assumption in general in the literature (e.g., Gabszewicz and Poddar (1997), Xu (2005), Murphy and Smeers (2005), Lagerlöf (2006)). We denote the cumulative distribution of $\alpha(\omega)$ by Φ whose support is on some interval $[\alpha_{min}, \alpha_{max}]$.

Dealing with an elastic demand (without VOLL pricing) instead of a fixed demand with VOLL pricing will slightly change the equilibrium conditions of the second stage game and the corresponding OPF problem for pool and bilateral markets. For inelastic demand in Section 4.3, we showed that the first and second stage equilibrium of bilateral and pool market models are equivalent. This result will also hold when demand is endogenous. Hence, we will continue this section with pool market model formulation.

4.4.1 Pool Market Model

Perfect Competition Equilibrium at Second Stage: When demand responds to prices, the corresponding Optimal Power Flow Problem (OPF) will slightly be different from the OPF Problem (4.1). For any $\omega \in \Omega$ and x,

$$\begin{aligned}
Z^*_{pool}(\omega, x) = \min_{\{y(\omega), f(\omega), d(\omega)\}} \quad & \sum_{g \in G} \sum_{i \in I_g} \sum_{k \in K_g} c^g_{ik} y^g_{ik}(\omega) - \sum_{n \in N} \int_0^{d_n(\omega)} P_n(\omega, s) ds, \\
s.t. \quad & \sum_{g \in G} \sum_{k \in K_g} y^g_{jk}(\omega) + f_j(\omega) \geq d_j(\omega) \qquad (p_j(\omega)) \quad \forall j \in \{N \cup I\} \\
& y(\omega) \text{ satisfy } Cons_Cap(\omega, x) \\
& f(\omega) \text{ satisfy } Cons_PTDF(\omega) \\
& y(\omega) \geq 0, \quad d(\omega) \geq 0,
\end{aligned} \tag{4.11}$$

where $d_n(\omega)$ is the decision variable representing endogenous demand in the state of the world ω and $\int_0^{d_n(\omega)} P_n(\omega, s) ds$ can be interpreted as the consumer's willingness to pay in the state of the world ω. The endogenous demand assumption results in the second stage equilibrium conditions in which the following conditions are different compared to the equilibrium conditions given in (4.2). For each $\omega \in \Omega$ and x,

$$\begin{aligned} 0 \le c_{ik}^{g} - p_i^*(\omega) + \beta_{ik}^{*g}(\omega) \quad &\perp \quad y_{ik}^{*g}(\omega) \ge 0 \quad \forall g \in G, i \in I_g, k \in K_g \\ 0 \le p_n^*(\omega) - P(\omega, d_n^*(\omega)) \quad &\perp \quad d_n^*(\omega) \ge 0 \quad \forall n \in N \\ 0 \le \sum_{g \in G} \sum_{k \in K_g} y_{jk}^{*g}(\omega) - f_j^*(\omega) - d_j^*(\omega) \quad &\perp \quad p_j^*(\omega) \ge 0 \quad j \in \{N \cup I\}. \end{aligned}$$

The above conditions imply that, at equilibrium, if any generator sells power at demand node n ($d_n^*(\omega) > 0$) then the market price at node n is equal to the price which consumers are willing to pay. Moreover, if any firm produces power at node i ($y_{ik}^{*g}(\omega) > 0$), then the market price at node i is equal to its marginal cost plus scarcity rent. Then for given x and $\omega \in \Omega$, the KKT conditions associated with a positive consumption and generation can be stated as:

$$\begin{aligned} p_n^*(\omega, x) \quad &= P(\omega, d_n^*(\omega, x)), \forall n \in N, \\ p_i^*(\omega, x) \quad &= c_{ik}^{g} + \beta_{ik}^{*g}(\omega, x), \forall g \in G, i \in I_g, k \in K_g. \end{aligned} \tag{4.12}$$

First Stage Behavior with Market Power: For given x^{*-g}, each firm's behavior and equilibrium conditions at first stage is the same. The profit function ($\Pi_{pool}^{g}(\omega, x^g, x^{-g})$) for firm $g \in G$ at each realization $\omega \in \Omega$ and the solution set $\chi_{pool}^{g}(x^{-g})$ can be formulated similar to the first stage profit function and the solution set given in Section 4.3.1.2.

4.4.2 Characterization of the Two-stage Game

In order to preserve analytical tractability as in Section 4.3.4, we again consider a simplified model without any network limits. Then the model reduces to a single node electricity market where all demand and supply is concentrated at one node ($d^*(\omega) = \sum_n d_n^*(\omega)$) and Assumption 4.3.3 *(i)* holds. Combining Assumption 4.3.3 *(i)* and equation (4.12), the KKT condition associated with a positive generation $y_k^*(\omega, x) > 0$ in a single node market can be stated, for given x and $\omega \in \Omega$, as

$$p^*(\omega, x) = P(\omega, d^*(\omega, x)) = c_k + \beta_k^*(\omega, x), \forall k \in K. \tag{4.13}$$

According to (4.13), consumers' willingness to pay is equal to a single market price paid to each firm when the spot market is cleared. Furthermore, each generating firm receives scarcity rent being equal to the market price minus its marginal cost. Thus, from now on $p^*(\omega, x)$ and $P(\omega, d^*(\omega, x))$ will be used interchangeably when we formulate the first stage problem of the firms.

Besides Assumption 4.3.3 *(i)*, we also make the following assumptions for the two-stage model with linear price-demand curve.

Assumption 4.4.1. *α is a continuous random variable with a cumulative distribution Φ and a continuous probability density function Φ' whose support is on some interval $[\alpha_{min}, \alpha_{max}]$.*

Assumption 4.4.2. *For at least one technology $k \in K$, it holds that*

$$\int_{\alpha_{min}}^{\alpha_{max}} P(s,0)\Phi^{'}(s)ds > c_k + \kappa_k.$$

When there is zero capacity ($\sum_{j=1}^{K} x_j = 0$), consumers are not supplied with any power ($d^*(\omega) = 0$) and are willing to pay $P(\omega,0)$ for the first unit of power at all realizations, which yields

$$\int_{\alpha_{min}}^{\alpha_{max}} P(s,0)\Phi^{'}(s)ds = E[\alpha(\omega)]$$

as the expected price that consumers are willing to pay for the first unit of power. According to Assumption 4.4.2, there exists at least one technology, say $k' \in K$, such that the expected price that consumers are willing to pay for the first unit of power from technology k' is higher than its cost of investment and operation. Therefore, firm k' will be able to increase its expected profit by investing at a positive level in technology k'; i.e., $x_{k'} > 0$. Thus, Assumption 4.4.2 guarantees that $\sum_{j=1}^{K} x_j^* > 0$.

Similar to the exogenous demand case, the expected profit function of firm $k \in K$ for given x_{-k} can be formulated in terms of its expected scarcity rent:

$$E_\omega[\Pi^k(\omega, x_k, x_{-k})] = (E_\omega[\beta_k^*(\omega, x_k, x_{-k})] - \kappa_k)x_k.$$

Next, we derive a closed form expression for the expected scarcity rent function for cases with symmetric and asymmetric firms and establish the characteristics of the expected scarcity rent and the expected profit functions for each firm. Different from the case with exogenous demand, the expected scarcity rent and the expected profit function of each firm are differentiable everywhere. On the other hand, similar to the case with exogenous demand, when firms are symmetric we show that the expected profit of each firm is strictly quasiconcave under logconcavity assumption of underlying probability density function and there exists a unique symmetric Nash equilibrium. In case of asymmetric firms, we can still show existence and uniqueness of equilibria under a stricter condition than logconcavity of probability density function.

4.4.2.1 Symmetric Firms

As mentioned in Section 4.3 for the exogenous demand case, the assumption of symmetric firms implies that the unit costs are the same for all firms; that is, $c_1 = c_2 = \ldots = c_K = c$ and $\kappa_1 = \kappa_2 = \cdots = \kappa_K = \kappa$. This yields an identical expected scarcity rent and profit function for all firms as shown in Proposition

4.4.3.

For given (x_k, x_{-k}), Figure 4.3 illustrates the competitive equilibrium in the spot market for a subset of realizations of the demand curve. As illustrated in the figure, there is a threshold value of intercept, say $\alpha(\widehat{\omega})$, which depends on (x_k, x_{-k}) and from which all firms' capacities are fully utilized. For realizations $\alpha(\omega) \leq \alpha(\widehat{\omega})$, firms' total available generation capacity is not fully utilized and the competitive market price is equal to the firms' marginal generating cost where the corresponding demand curve crosses the supply curve. According to (4.13), this yields zero scarcity rent for all firms. For realizations $\alpha(\omega) > \alpha(\widehat{\omega})$, firms' total available generation capacity is fully utilized (e.g., $d^*(\omega) = X_{K-1}(k) + x_k$ where $X_{K-1}(k) = \sum_{j=1, j\neq k}^{K} x_j$) and the market price is higher than the firms' marginal generating cost, which yields a positive scarcity rent equal to $P(\omega, X_{K-1}(k) + x_k) - c$ for all firms.

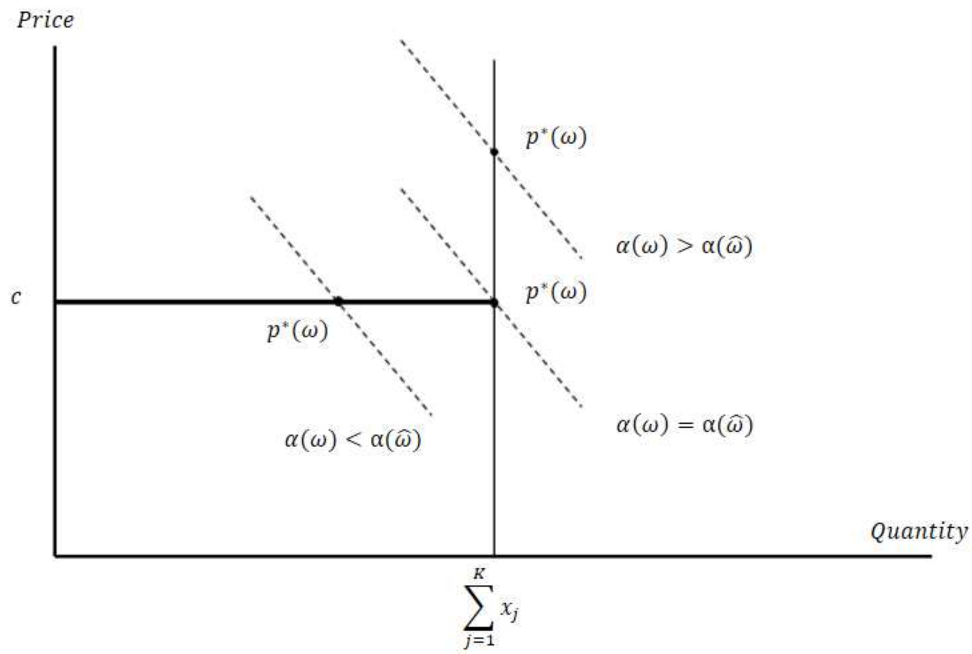


Figure 4.3: Illustration of a single-node market with symmetric firms and linear demand curve with random intercept for given x

As mentioned before, the value of $\alpha(\widehat{\omega})$ depends on (x_k, x_{-k}) which can be easily derived from Figure 4.3 as:

$$\alpha(\widehat{\omega}) = c + \gamma \cdot (X_{K-1}(k) + x_k).$$

Next, we derive the closed form expression of firm k's expected scarcity rent function for given x_{-k}.

Proposition 4.4.3. *Under Assumption 4.4.1, for given x_{-k}, the expected scarcity rent of firm $k \in K$ can* *be formulated as follows:*

$$E_\omega[\beta_k^*(\omega, x_k, x_{-k})] = \int_{\alpha(\widehat{\omega})}^{\alpha_{max}} (1-\Phi(s))ds, \tag{4.14}$$

where $\alpha(\widehat{\omega}) = c + \gamma \cdot (X_{K-1}(k) + x_k)$.

Proof. See Appendix 4.A.

In the following lemmas, we establish the differentiability of both expected scarcity rent and expected profit functions with respect to x_k which also imply their continuity. Then, we show in Theorem 4.4.5 that $E_\omega[\Pi^k(\omega, x_k, x_{-k})]$ is strictly logconcave under certain conditions on the support of cumulative distribution function Φ.

Lemma 4.4.4. *Let* Φ *be a differentiable function on its support. For given* x_{-k}*, the expected scarcity rent of each firm* $k \in K$ *given in (4.14) is differentiable and nonincreasing w.r.t.* $x_k \in \mathscr{R}_+$ *with:*

$$\frac{\partial E_\omega[\beta_k^*(\omega, x_k, x_{-k})]}{\partial x_k} = -\gamma \cdot (1 - \Phi(\alpha(\widehat{\omega})) \leq 0. \tag{4.15}$$

Moreover, $E_\omega[\beta_k^*(\omega, x_k, x_{-k})]$ *is a convex function of* $x_k \in \mathscr{R}_+$.

Proof. See Appendix 4.A.

Theorem 4.4.5. *Let* Φ *be a differentiable function on its support. For given* $k \in K$ *and* x_{-k},

(i) $E_\omega[\Pi^k(\omega, x_k, x_{-k})]$ *is differentiable w.r.t.* $x_k \in \mathscr{R}_+$*, and*

(ii) if the probability density function Φ' *is logconcave, then* $E_\omega[\Pi^k(\omega, x_k, x_{-k})]$ *is a strictly logconcave function of* $x_k \in \mathscr{R}_+$.

Proof. See Appendix 4.A.

As in the exogenous demand case, we prove in the next lemmas that the strategy space of each firm is bounded and convex. Then in Theorem 4.4.7, we show existence and uniqueness of Nash equilibrium.

Lemma 4.4.6. *The strategy space of each firm* $k \in K$ *is* $S_k := [0, \frac{\alpha_{max} - c}{\gamma}]$ *which is nonempty, compact, and convex.*

Proof. See Appendix 4.A.

Theorem 4.4.7 (Existence and Uniqueness). *Let Φ be a differentiable function on its support and the probability density function Φ' be logconcave and Assumption 4.4.2 holds. Then for the game defined in Section 4.4.1 with symmetric firms and endogenous demand curve with random intercept, there exists a unique symmetric Nash equilibrium, $x^* = x_1^* = \ldots = x_K^*$, which satisfies*

$$x^* = \frac{\int_{\alpha(\widehat{\omega})}^{\alpha_{max}} (1 - \Phi(\omega)) d\omega - \kappa}{\gamma \cdot (1 - \Phi(\alpha(\widehat{\omega})))}, \tag{4.16}$$

and $x^* > 0$ where $\widehat{\omega}$ is such that $\alpha(\widehat{\omega}) = c + K\gamma x^*$.

Proof. See Appendix 4.A.

4.4.2.2 Asymmetric Firms

For given (x_k, x_{-k}), Figure 4.4 illustrates the competitive equilibrium in the spot market for a subset of realizations of demand curve. In comparison to Figure 4.3 of the symmetric case, the supply curve of asymmetric generators is a piecewise constant function having more than one piece. Thus, there are multiple threshold values of intercept, $\alpha(\underline{\omega}_1) < \alpha(\overline{\omega}_1) < \alpha(\underline{\omega}_2) < \alpha(\overline{\omega}_2) < \ldots < \alpha(\underline{\omega}_K) < \alpha(\overline{\omega}_K)$, at which the market price changes as explained below. For instance, let's take $k = 1$ in Figure 4.4 and consider the set of realizations $[\alpha(\underline{\omega}_1), \alpha(\overline{\omega}_1)]$ and $[\alpha(\overline{\omega}_1), \alpha(\underline{\omega}_2)]$ at which firm 1 is the marginal unit and the other firms do not generate at all. For given x_1, the market price is equal to the sum of the marginal cost and the scarcity rent of firm 1 (c_1 and β_1^*) for the realizations within these sets:

(a) for $\alpha(\omega) \in [\alpha(\underline{\omega}_1), \alpha(\overline{\omega}_1))$, the capacity of firm 1 is not binding (i.e., $d(\omega) < x_1$), hence the market price is equal to c_1 and $\beta_1^*(\omega, x_1, x_{-1}) = 0$,

(b) for $\alpha(\omega) \in [\alpha(\overline{\omega}_1), \alpha(\underline{\omega}_2)]$, the capacity of firm 1 is binding (i.e.,$d(\omega) = x_1$) and the market price is equal to $P(\omega, x_1)$ and $\beta_1^*(\omega, x_1, x_{-1}) = P(\omega, x_1) - c_1$.

Similarly, let's consider the set of realizations $[\alpha(\underline{\omega}_2), \alpha(\overline{\omega}_2)]$ and $[\alpha(\overline{\omega}_2), \alpha(\underline{\omega}_3)]$ at which firm 1's capacity is fully utilized and firm 2 is the marginal unit. The other firms having higher marginal costs do not generate at all. Then for given x_1, the market price is equal to the sum of the marginal cost and scarcity rent of firm 2 (c_2 and β_2^*) for the realizations within these sets:

(c) for $\alpha(\omega) \in [\alpha(\underline{\omega}_2), \alpha(\overline{\omega}_2))$, the capacity of firm 2 is not binding (i.e., $x_1 < d(\omega) < x_1 + x_2$) hence the market price is equal to c_2 and $\beta_1^*(\omega, x_1, x_{-1}) = c_2 - c_1$,

(d) for $\alpha(\omega) \in [\alpha(\overline{\omega}_2), \alpha(\underline{\omega}_3)]$, the capacity of firm 2 is binding (i.e., $d(\omega) = x_1 + x_2$) and the market price is equal to $P(\omega, x_1 + x_2)$ and $\beta_1^*(\omega, x_1, x_{-1}) = P(\omega, x_1 + x_2) - c_1$.

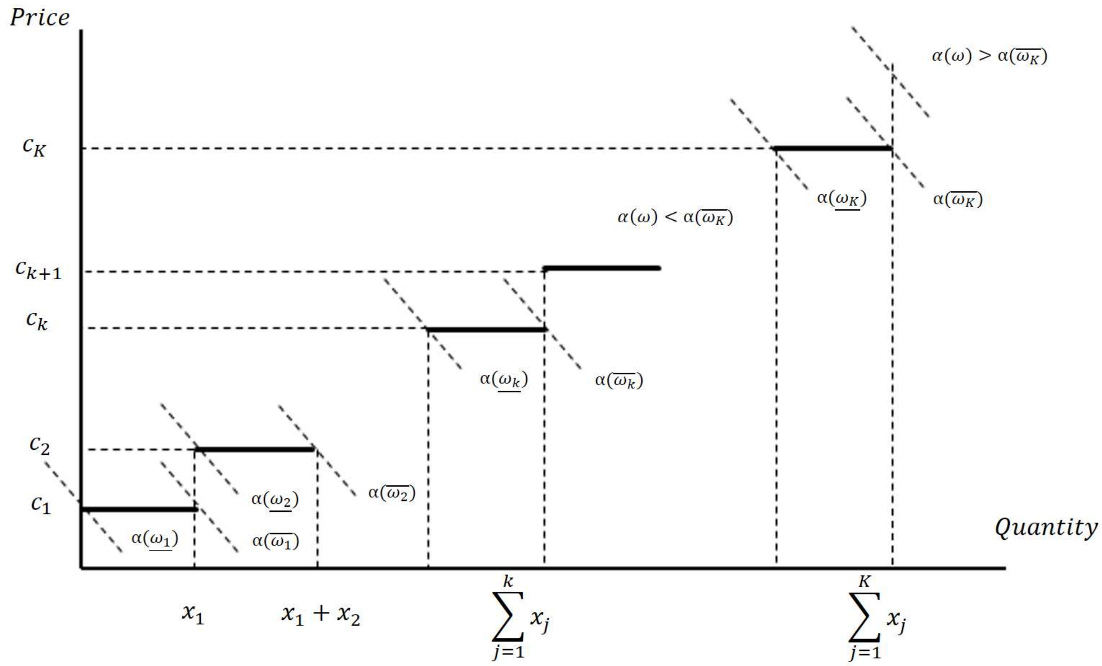


Figure 4.4: Illustration of a single-node market with asymmetric firms and linear demand curve with random intercept for given x

Next we will generalize the formulation of $\beta_k^*(\omega, x_k, x_{-k})$ for each k and $\alpha(\omega) \in [\alpha_{min}, \alpha_{max}]$. The value of $\beta_k^*(\omega, x_k, x_{-k})$ depends on the type of firm being the marginal unit at each $\alpha(\omega)$ as explained above. At a realization $\alpha(\omega)$, if firms $1, 2, \ldots, m$ are generating, then firm m is called the marginal unit since it is the most expensive firm setting the market price. Furthermore, the scarcity rent of firm k is equal to the market price minus the marginal cost of firm k. For a given k, we will specify a range of $\alpha(\omega)$ where the marginal unit is firm m with a higher marginal cost ($m \geq k$). Otherwise when the marginal unit is with lower marginal cost ($m < k$), firm k is not generating at all and its scarcity rent is zero. Note that in the above example, (a) and (b) show the situation where $k = m = 1$ whereas (c) and (d) show the situation where $k = 1$ and $m = 2$.

For $k \leq m \leq K-1$, let $\overline{\Omega}_m := [\alpha(\overline{\omega}_m), \alpha(\underline{\omega}_{m+1})]$ denote the set of realizations where the capacity of marginal unit m is binding; that is $d(\omega) = \sum_{j=1}^{m} x_j$. Similarly, $\overline{\Omega}_K := [\alpha(\overline{\omega}_K), \alpha_{max}]$ is the set of realizations where all firms' capacities are binding. In addition, for $k < m \leq K$, let $\underline{\Omega}_m := [\alpha(\underline{\omega}_m), \alpha(\overline{\omega}_m))$ denote the set of realizations where the marginal unit m is generating but its capacity is not fully utilized; that is $\sum_{j=1}^{m-1} x_j < d(\omega) < \sum_{j=1}^{m} x_j$. Then for each firm $k \in K$, the following holds:

(i) For $\alpha_{min} \leq \alpha(\omega) < \alpha(\overline{\omega}_k)$, the capacity of firm k is not fully utilized, hence $\beta_k^*(\omega, x_k, x_{-k}) = 0$.

(ii) For $\alpha(\omega) \in \overline{\Omega}_m (m \geq k)$, firm k and marginal unit m generate at full capacity and market price is equal to $P(\omega, \sum_{j=1}^{m} x_j)$. Thus, $\beta_k^*(\omega, x_k, x_{-k}) = P(\omega, \sum_{j=1}^{m} x_j) - c_k \geq 0$.

(iii) For $\alpha(\omega) \in \underline{\Omega}_m (m \geq k+1)$, firm k generates at full capacity but marginal unit m generates less than its capacity. Then the market price is equal to the marginal cost of unit m. Thus, $P(\omega, d(\omega)) = c_m$ and $\beta_k^*(\omega, x_k, x_{-k}) = c_m - c_k > 0$.

Recall that for given x_{-k}, we denote the total capacity of the $(m-1)$ cheapest firms excluding firm k in the market as $X_{m-1}(k) = \sum_{j=1, j \neq k}^{m} x_j$, for $k \leq m \leq K$. Then $X_{m-1}(k) + x_k$ is the total capacity of the m cheapest firms including firm k in the market. Then by using (i)-(iii) above and for given x_{-k}, we get:

$$
\begin{aligned}
\beta_k^*(\omega, x_k, x_{-k}) = \quad & 0 I_{\{\alpha_{min} \leq \alpha(\omega) < \alpha(\overline{\omega}_k)\}} \\
& + \sum_{m=k}^{K-1} (P(\omega, X_{m-1}(k) + x_k) - c_k) I_{\{\alpha(\overline{\omega}_m) \leq \alpha(\omega) \leq \alpha(\underline{\omega}_{m+1})\}} \\
& + (P(\omega, X_{K-1}(k) + x_k) - c_k) I_{\{\alpha(\overline{\omega}_K) \leq \alpha(\omega) \leq \alpha_{max}\}} \\
& + \sum_{m=k+1}^{K} (c_m - c_k) I_{\{\alpha(\underline{\omega}_m) < \alpha(\omega) < \alpha(\overline{\omega}_m)\}}.
\end{aligned}
$$

Then $E_\omega[\beta_k^*(\omega, x_k, x_{-k})]$ can be formulated as follows:

$$
\begin{aligned}
E_\omega[\beta_k^*(\omega, x_k, x_{-k})] = \quad & \sum_{m=k}^{K-1} \int_{\alpha(\overline{\omega}_m)}^{\alpha(\underline{\omega}_{m+1})} (P(s, X_{m-1}(k) + x_k) - c_k) \Phi'(s) ds \\
& + \int_{\alpha(\overline{\omega}_K)}^{\alpha_{max}} (P(s, X_{K-1}(k) + x_k) - c_k) \Phi'(s) ds \\
& + \sum_{m=k+1}^{K} \int_{\alpha(\underline{\omega}_m)}^{\alpha(\overline{\omega}_m)} (c_m - c_k) \Phi'(s) ds
\end{aligned}
\tag{4.17}
$$

By using the explicit formulation of the linear price-demand function, (4.17) can further be simplified as a closed form expression as given in the next proposition.

Proposition 4.4.8. *Under Assumption 4.4.1, for given x_k, the expected scarcity rent of firm $k \in K$ can be formulated in a closed form expression as follows:*

$$
E_\omega[\beta_k^*(\omega, x_k, x_{-k})] = \quad = (c_K - c_k) + \int_{\alpha(\overline{\omega}_K)}^{\alpha_{max}} (1 - \Phi(s)) ds - \sum_{m=k}^{K-1} \int_{\alpha(\overline{\omega}_m)}^{\alpha(\underline{\omega}_{m+1})} \Phi(s) ds, \tag{4.18}
$$

where, for $m = k, \ldots, K-1$, the threshold values $\alpha(\overline{\omega}_m)$ and $\alpha(\underline{\omega}_{m+1})$ depend on the capacities of firms *generating at* $\overline{\omega}_m$ *and* $\underline{\omega}_{m+1}$*, respectively and they can be easily derived from Figure 4.4:*

$$\alpha(\overline{\omega}_m) = c_m + \gamma \cdot (X_{m-1}(k) + x_k) \text{ and } \alpha(\underline{\omega}_{m+1}) = c_{m+1} + \gamma \cdot (X_{m-1}(k) + x_k), \forall m \geq k. \tag{4.19}$$

Proof. See Appendix 4.A.

Note that when firms are symmetric $c_K = c_k = c$ and $\alpha(\overline{\omega}_m) = \alpha(\underline{\omega}_{m+1}), \forall m$. Then, (4.18) is reduced to (4.14). The expected scarcity rent function of each firm given in (4.18) is differentiable and decreasing as shown in the next lemma.

Lemma 4.4.9. *Let* Φ *be a differentiable function on its support. For given* x_{-k}*, the expected scarcity rent of each firm* $k \in K$ *given in* (4.18) *is differentiable w.r.t.* $x_k \in \mathscr{R}_+$*. Moreover,* (4.18) *is a decreasing function of* $x_k \in \mathscr{R}_+$.

Proof. See Appendix 4.A.

Next, we can show strict concavity of the expected profit function of each firm under a sufficient condition satisfied by the probability distribution function.

Theorem 4.4.10. *Let* Φ *be a differentiable function on its support. For given* x_{-k}*,*

(i) the expected profit function of each firm k is differentiable w.r.t. $x_k \in \mathscr{R}_+$*, and*

(ii) the expected profit function of each firm k is (strictly) concave if $-\Phi'(s+C) - s \cdot \Phi''(s+C)(<) \leq 0$*, for all* $s, C \in \mathscr{R}_+$*, where C is some constant.*

Proof. See Appendix 4.A.

The condition stated in Theorem 4.4.10 *(ii)* is again similar to the standard assumption used in the literature (see Sherali et al. (1983), Wolf and Smeers (1997), Grimm and Zoettl (2008), and Xu (2005)). Rosen (1965) shows that there is a unique equilibrium point for every strictly concave game. We next show that the strategy space of each firm is bounded and convex. Finally, in Theorem 4.4.12, we conclude with existence and uniqueness of a Nash equilibrium for the two-stage game with asymmetric firms by using the result of Rosen (1965).

Lemma 4.4.11. *The strategy space of each firm* $k \in K$ *is* $S_k := [0, \frac{\alpha_{max} - c_k}{\gamma}]$ *which is nonempty, compact, and convex.*

Proof. See Appendix 4.A.

Theorem 4.4.12. ***[Existence and Uniqueness]*** *Let Φ be a differentiable function on its support and $-\Phi'(s+C) - s \cdot \Phi''(s+C) < 0$, for all $s, C \in \mathscr{R}_+$, where C is some constant. Then, under Assumption 4.4.2, for the game with asymmetric firms and endogenous demand curve with random intercept, there exists a unique Nash equilibrium.*

Proof. See Appendix 4.A.

4.5 Conclusions

In this chapter, we establish sufficient conditions which guarantee existence and uniqueness of equilibria in oligopolistic electricity markets where strategic electricity generators anticipate perfectly competitive spot market outcomes with demand uncertainty while choosing their capacities and their power generation is dispatched after the level of demand is realized. In case of symmetric firms, we show that a large class of continuous probability distributions guarantee uniqueness of equilibrium for the two-stage game. In case of asymmetric firms, equilibrium may not exist since the first-stage payoff functions of firms do not, in general, satisfy generalized concavity when demand is exogenous. When demand is endogenous, a condition on probability distribution function, which is similar to the standard assumption used in the literature for inverse demand curve, is sufficient to guarantee uniqueness of the equilibrium.

In general, two-stage closed loop models with strategic firms are nonconvex problems and examples have been posed in the literature that have multiple or no equilibria. Therefore, it is of interest to users of such closed loop models to know general sets of conditions, as we define here, under which the existence of a unique equilibrium is guaranteed. Availability of such conditions will enhance the value of these closed loop models for their implementation for policy and market intelligence purposes.

As a final remark, in a closed loop game with two asymmetric firms facing an elastic demand, we numerically computed the expected profit function of each firm under various logconcave probability distributions of random intercept (e.g., uniform, normal, exponential, Weibull, gamma, beta) and we observed that each firm's expected profit is strictly quasiconcave in all the cases. Thus, we conjecture that the logconcavity of probability distributions may also be sufficient to guarantee the existence of a unique equilibrium for closed loop games with asymmetric firms when demand is endogenous. However, we could not establish a theoretical proof for this conjecture which remains to be a topic for future research.

4.A Proofs of Propositions, Lemmas, and Theorems

Proof of Proposition 4.3.1. First, we look at the profit of firm $g \in G$ at realization $\omega \in \Omega$ in a bilateral market at a given x^{-g}:

$$\begin{aligned}\Pi^g_{bilateral}(\omega,x^g,x^{-g}) = & \sum_{n\in N}[p^{*b}_n(\omega,x^g,x^{-g}) - v^{*b}_n(\omega,x^g,x^{-g})]s^{*bg}_n(\omega,x^g,x^{-g}) \\ & - \sum_{i\in I_g}\sum_{k\in K_g}[c^g_{ik} - v^{*b}_i(\omega,x^g,x^{-g})]y^{*bg}_{ik}(\omega,x^g,x^{-g}) - \sum_{i\in I_g}\sum_{k\in K_g}\kappa_k x^g_{ik}.\end{aligned}$$

From the KKT conditions in (4.4), we have:
If $y^{*bg}_{ik}(\omega,x^g,x^{-g}) > 0$, then $v^{*b}_i(\omega,x^g,x^{-g}) = c^g_{ik} + \beta^{*bg}_{ik}(\omega,x^g,x^{-g}) - \rho^{*b}_g(\omega,x^g,x^{-g})$.
If $s^{*bg}_n(\omega,x^g,x^{-g}) > 0$, then $v^{*b}_n(\omega,x^g,x^{-g}) = p^{*b}_n(\omega,x^g,x^{-g}) - \rho^{*b}_g(\omega,x^g,x^{-g})$.
Then by utilizing these together with the third constraint of OPF problem (4.3) we can simplify the profit function to:

$$\Pi^g_{bilateral}(\omega,x^g,x^{-g}) = \sum_{i\in I_g}\sum_{k\in K_g}\beta^{*bg}_{ik}(\omega,x^g,x^{-g})y^{*bg}_{ik}(\omega,x^g,x^{-g}) - \sum_{i\in I_g}\sum_{k\in K_g}\kappa_k x^g_{ik}.$$

We also know that if $\beta^{*bg}_{ik}(\omega,x^g,x^{-g}) > 0$, then $y^{*bg}_{ik}(\omega,x^g,x^{-g}) = x^g_{ik}$; hence the profit function of the bilateral market model becomes:

$$\Pi^g_{bilateral}(\omega,x^g,x^{-g}) = \sum_{i\in I_g}\sum_{k\in K_g}\beta^{*bg}_{ik}(\omega,x^g,x^{-g})x^g_{ik} - \sum_{i\in I_g}\sum_{k\in K_g}\kappa_k x^g_{ik}.$$

Now, we look at the profit of firm $g \in G$ at realization $\omega \in \Omega$ in pool market at a given x:

$$\Pi^g_{pool}(\omega,x^g,x^{-g}) = \sum_{i\in I_g}\sum_{k\in K_g}(p^{*p}_i(\omega,x^g,x^{-g}) - c^g_{ik})y^{*pg}_{ik}(\omega,x^g,x^{-g}) - \sum_{i\in I_g}\sum_{k\in K_g}\kappa_k x^g_{ik}.$$

From the KKT conditions in (4.2), we have:
If $y^{*pg}_{ik}(\omega,x^g,x^{-g}) > 0$, then $p^{*p}_i(\omega,x^g,x^{-g}) = c^g_{ik} + \beta^{*pg}_{ik}(\omega,x^g,x^{-g})$.
Using this, we can again simplify the profit function to :

$$\begin{aligned}\Pi^g_{pool}(\omega,x^g,x^{-g}) &= \sum_{i\in I_g}\sum_{k\in K_g}\beta^{*pg}_{ik}(\omega,x^g,x^{-g})y^{*pg}_{ik}(\omega,x^g,x^{-g}) - \sum_{i\in I_g}\sum_{k\in K_g}\kappa_k x^g_{ik} \\ &= \sum_{i\in I_g}\sum_{k\in K_g}\beta^{*pg}_{ik}(\omega,x^g,x^{-g})x^g_{ik} - \sum_{i\in I_g}\sum_{k\in K_g}\kappa_k x^g_{ik}.\end{aligned}$$

The last equality follows with a similar reasoning used for the bilateral case.

The result given in Proposition 4.2.1 (i) indicates that at any given (x^g,x^{-g}), $\beta^{*pg}_{ik}(\omega,x^g,x^{-g}) = \beta^{*bg}_{ik}(\omega,x^g,x^{-g})$. Therefore, $\Pi^g_{pool}(\omega,x^g,x^{-g}) = \Pi^g_{bilateral}(\omega,x^g,x^{-g})$ which in turn implies $E_\omega[\Pi^g_{pool}(\omega,x^g,x^{-g})] = E_\omega[\Pi^g_{bilateral}(\omega,x^g,x^{-g})]$. Thus, for given x^{-g}, if there exists a $x^{*g} \in \chi_{bilateral}(x^{-g})$ then it also holds that

$x^{*g} \in \chi_{pool}(x^{-g})$ for all $g \in G$ and vice versa. Hence, the optimal solution sets of the upper level problems in both market structures are identical which implies both markets yield the same first stage equilibria. ∎

Proof of Proposition 4.3.4. We know that $\beta_k^*(\omega, x_k, x_{-k}) = p^*(\omega, x_k, x_{-k}) - c_k$. Thus, for given $\{X_{m-1}(k)\}_{m=k}^K$, the scarcity rent paid to firm $k \in K$ at realization $\omega\Omega$ can be formulated as follows:

$$\begin{aligned}
\beta_k^*(\omega, x_k, x_{-k}) = \quad & 0 I_{\{\underline{D} \leq D \leq X_{k-1}(k)+x_k\}} \\
+ \quad & \sum_{m=k}^{K-1} (c_{m+1} - c_k) I_{\{X_{m-1}(k)+x_k < D \leq X_m(k)+x_k\}} \\
+ \quad & (VOLL - c_k) I_{\{X_{K-1}(k)+x_k < D \leq \overline{D}\}}.
\end{aligned}$$

Then calculation of $E_\omega[\beta_k^*(\omega, x_k, x_{-k})]$ will yield the following:

$$\begin{aligned}
E_\omega[\beta_k^*(\omega, x_k, x_{-k})] &= \sum_{m=k}^{K-1} \int_{X_{m-1}(k)+x_k}^{X_m(k)+x_k} (c_{m+1} - c_k) d(\Psi(s)) + \int_{X_{K-1}(k)+x_k}^{\overline{D}} (VOLL - c_k) d(\Psi(s)) \\
&= \sum_{m=k}^{K-1} (c_{m+1} - c_k)[\Psi(X_m(k) + x_k) - \Psi(X_{m-1}(k) + x_k)] \\
&\quad + (VOLL - c_k)(1 - \Psi(X_{K-1}(k) + x_k)).
\end{aligned} \tag{4.20}$$

The expression (4.20) can be simplified to:

$E_\omega[\beta_k^*(\omega, x_k, x_{-k})] = VOLL - c_k - (VOLL - c_K)\Psi(X_{K-1}(k) + x_k) - \sum_{m=k}^{K-1} ((c_{m+1} - c_m)\Psi(X_{m-1}(k) + x_k))$. ∎

Proof of Lemma 4.3.5.

(i) From (4.6), it is easy to see that $E_\omega[\beta_k^*(\omega, x_k, x_{-k})]$ is a continuous function of $(x_k, x_{-k}) \in \mathscr{R}_+^{\mathscr{K}}$ since Ψ is the cumulative distribution function of continuous random demand.

(ii) We know that Ψ is a nondecreasing function of its argument. Fix a k, then for all $0 \leq a < b$:

$$\begin{aligned}
0 \quad & \leq \Psi(X_l(k) + a) \leq \Psi(X_l(k) + b) \leq 1,\ k - 1 \leq l \leq K - 1 \text{ which implies} \\
0 \quad & \leq E_\omega[\beta_k^*(\omega, b, x_{-k})] = VOLL - c_k - (VOLL - c_K)\Psi(X_{K-1}(k) + b) - \sum_{m=k}^{K-1} (c_{m+1} - c_m)\Psi(X_{m-1}(k) + b) \\
& \leq VOLL - c_k - (VOLL - c_K)\Psi(X_{K-1}(k) + a) - \sum_{m=k}^{K-1} (c_{m+1} - c_m)\Psi(X_{m-1}(k) + a) = E_\omega[\beta_k^*(\omega, a, x_{-k})]. \blacksquare
\end{aligned}$$

Proof of Lemma 4.3.6. By Lemma 4.3.5, we know that $E_\omega[\beta_k^*(\omega, x_k, x_{-k})]$ in (4.7) is a continuous function on $[0, \infty]$ where

$$E_\omega[\beta_k^*(\omega, x_k, x_{-k})] = \begin{cases} VOLL - c, & 0 \leq x_k < x_k^{min} \\ \\ (VOLL - c)(1 - \Psi(x_k + X_{K-1}(k)), & x_k^{min} \leq x_k < x_k^{max} \\ \\ 0, & \text{otherwise.} \end{cases} \tag{4.21}$$

Therefore, whenever Ψ is differentiable, then $E_\omega[\beta_k^*(\omega, x_k, x_{-k})]$ is also differentiable w.r.t. $x_k \in \mathscr{R}_+$ except at the breakpoints given in (4.21):

$$\frac{\partial E_\omega[\beta_k^*(\omega, x_k, x_{-k})]}{\partial x_k} = \begin{cases} 0, & x_k \in [0,\, x_k^{min}) \cup [x_k^{max},\, \infty) \\ \\ -(VOLL - c)\Psi'(x_k + X_{K-1}(k)), & x_k \in [x_k^{min},\, x_k^{max}). \blacksquare \end{cases} \tag{4.22}$$

Proof of Theorem 4.3.8. Since $E_\omega[\Pi^k(\omega, x_k, x_{-k})] = (E_\omega[\beta_k^*(\omega, x_k, x_{-k})] - \kappa)x_k$, *(i)* and *(ii)* directly follow from Lemma 4.3.5 *(i)* and Lemma 4.3.6, respectively.

(iii) $E_\omega[\Pi^k(\omega, x_k, x_{-k})]$ is continuous and differentiable w.r.t. x_k on $S := (x_k^{min}, x_k^{max})$. Therefore, we know that it is also (strictly) concave on S if and only if

$$\begin{aligned} & \frac{\partial^2 E_\omega[\Pi^k(\omega, x_k, x_{-k}))]}{\partial x_k^2} \; (<) \leq 0, && \forall x_k \in S \\ \Leftrightarrow \quad & \frac{\partial^2 E_\omega[\beta_k^*(\omega, x_k, x_{-k}))]}{\partial x_k^2} x_k + 2\frac{\partial E_\omega[\beta_k^*(\omega, x_k, x_{-k}))]}{\partial x_k} \; (<) \leq 0, && \forall x_k \in S \\ \Leftrightarrow \quad & -\Psi''(x_k + X_{K-1}(k))x_k - 2\Psi'(x_k + X_{K-1}(k)) \; (<) \leq 0, && \forall x_k \in S. \end{aligned}$$

Since $\Psi'(x_k + X_{K-1}(k))$ is nonnegative, the last inequality holds iff $-\Psi''(x_k + X_{K-1}(k))x_k - \Psi'(x_k + X_{K-1}(k))\; (<) \leq 0$.

Outside of S, $E_\omega[\Pi^k(\omega, x_k, x_{-k})]$ is equal to $(VOLL - c - \kappa)x_k$ on $(0, x_k^{min})$ and $-\kappa x_k$ on (x_k^{max}, ∞) which are increasing and decreasing linear functions, respectively. Therefore, whenever the stated condition holds, $E_\omega[\Pi^k(\omega, x_k, x_{-k})]$ is a (strictly) quasiconcave function of x_k on $\mathscr{R}_+$. $\blacksquare$

Proof of Theorem 4.3.10. $E_\omega[\Pi^k(\omega, x_k, x_{-k})]$ is equal to $(VOLL - c - \kappa)x_k$ on $[0, x_k^{min})$ and $-\kappa x_k$ on (x_k^{max}, ∞) which are increasing and decreasing linear functions, respectively. Therefore, if $E_\omega[\Pi^k(\omega, x_k, x_{-k})]$ is strictly quasiconcave on $[x_k^{min}, x_k^{max}]$, it is also strictly quasiconcave on $\mathscr{R}_+$, which we prove next.

We know that

$$\frac{\partial E_\omega[\Pi^k(\omega,x_k,x_{-k})]}{\partial x_k} = \frac{\partial E_\omega[\beta_k^*(\omega,x_k,x_{-k})]}{\partial x_k} x_k + E_\omega[\beta_k^*(\omega,x_k,x_{-k})] - \kappa,$$

which can be formulated explicitly by using (4.21) and (4.22):

$$\frac{\partial E_\omega[\Pi^k(\omega,x_k,x_{-k})]}{\partial x_k} = \begin{cases} VOLL - c - \kappa, & x_k \in [0, x_k^{min}) \\ (VOLL-c)[1-\Psi(x_k+X_{K-1}(k)) - \Psi'(x_k+X_{K-1}(k))x_k] - \kappa, & x_k \in [x_k^{min}, x_k^{max}) \\ -\kappa, & x_k \in [x_k^{max}, \infty). \end{cases}$$

– Then for $x_k \in (x_k^{min}, x_k^{max})$,

$$\frac{\partial E_\omega[\Pi^k(\omega,x_k,x_{-k})]}{\partial x_k} = -(VOLL-c)\Psi'(x_k+X_{K-1}(k))x_k + (VOLL-c)(1-\Psi(x_k+X_{K-1}(k))) - \kappa,$$

is the gradient of $E_\omega[\Pi^k(\omega,x_k^{min},x_{-k})]$.

– For the breakpoints $x_k \in \{x_k^{min}, x_k^{max}\}$,

$$\begin{aligned} \frac{\partial E_\omega[\Pi^k(\omega,x_k^{min},x_{-k})]}{\partial x_k} &:= \quad [VOLL-c-\kappa, \quad (VOLL-c)(1-\Psi'(\underline{D})x_k^{min}) - \kappa], \\ \frac{\partial E_\omega[\Pi^k(\omega,x_k^{max},x_{-k})]}{\partial x_k} &:= \quad [-\kappa, \quad -(VOLL-c)\Psi'(\overline{D})x_k^{max} - \kappa] < 0. \end{aligned} \tag{4.23}$$

are the subdifferentials of $E_\omega[\Pi^k(\omega,x_k,x_{-k})]$ at $x_k = x_k^{min}$ and $x_k = x_k^{max}$, respectively.

Furthermore, note that since the hazard function of demand is monotone increasing:

(i) $1 - \dfrac{\Psi'(x_k+X_{K-1}(k))}{(1-\Psi(x_k+X_{K-1}(k)))} x_k$ is a decreasing function of x_k, and

(ii) $\dfrac{\kappa}{(VOLL-c)(1-\Psi(x_k+X_{K-1}(k)))}$ is a nondecreasing function of x_k.

Next, we differentiate the three only possible cases :

Case (1) $x_k^{min} > 0$: We know that $E_\omega[\Pi^k(\omega,x_k,x_{-k})]$ is continuous, and it is an increasing function on $[0,x_k^{min})$ and a decreasing function on (x_k^{max},∞). Hence, there exists an $\tilde{x}_k \in [x_k^{min}, x_k^{max})$ such that

$$0 \in \frac{\partial E_\omega[\Pi^k(\omega,\tilde{x}_k,x_{-k})]}{\partial x_k}.$$

Note that $\tilde{x}_k \neq x_k^{max}$ since $\dfrac{\partial E_\omega[\Pi^k(\omega, x_k^{max}, x_{-k})]}{\partial x_k} < 0$ from (4.23). For $\tilde{x}_k$, the following holds:

$$-(VOLL-c)\Psi^{'}(\tilde{x}_k + X_{K-1}(k))\tilde{x}_k + (VOLL-c)(1-\Psi(\tilde{x}_k + X_{K-1}(k))) = \kappa,$$

$$\text{which implies } 1 - \frac{\Psi^{'}(\tilde{x}_k + X_{K-1}(k))}{(1-\Psi(\tilde{x}_k + X_{K-1}(k)))}\tilde{x}_k = \frac{\kappa}{(VOLL-c)(1-\Psi(\tilde{x}_k + X_{K-1}(k)))}. \tag{4.24}$$

Next we distinguish between the cases $\tilde{x}_k = x_k^{min}$ and $\tilde{x}_k > x_k^{min}$:

(1a) When $\tilde{x}_k = x_k^{min}$, by utilizing *(i)* and *(ii)*, for $x_k^{min} = \tilde{x}_k < x_k < x_k^{max}$,

$$\begin{aligned} 1 - \frac{\Psi^{'}(x_k + X_{K-1}(k))}{(1-\Psi(x_k + X_{K-1}(k)))}x_k \quad &< 1 - \frac{\Psi^{'}(\tilde{x}_k + X_{K-1}(k))}{(1-\Psi(\tilde{x}_k + X_{K-1}(k)))}\tilde{x}_k = \frac{\kappa}{(VOLL-c)(1-\Psi(\tilde{x}_k + X_{K-1}(k)))} \\ &\leq \frac{\kappa}{(VOLL-c)(1-\Psi(x_k + X_{K-1}(k)))}. \end{aligned}$$

Consequently,

$$-(VOLL-c)\Psi^{'}(x_k + X_{K-1}(k))x_k + (VOLL-c)(1-\Psi(x_k + X_{K-1}(k))) < \kappa,$$

which implies that

$$\frac{\partial E_\omega[\Pi^k(\omega, x_k, x_{-k})]}{\partial x_k} < 0.$$

Thus, $E_\omega[\Pi^k(\omega, x_k, x_{-k})]$ is a decreasing function and hence strictly quasiconcave on $[x_k^{min}, x_k^{max}]$.

(1b) When $\tilde{x}_k > x_k^{min}$, again by utilizing *(i)* and *(ii)* for $x_k^{min} < x_k < \tilde{x}_k$,

$$\begin{aligned} 1 - \frac{\Psi^{'}(x_k + X_{K-1}(k))}{(1-\Psi(x_k + X_{K-1}(k)))}x_k \quad &> 1 - \frac{\Psi^{'}(\tilde{x}_k + X_{K-1}(k))}{(1-\Psi(\tilde{x}_k + X_{K-1}(k)))}\tilde{x}_k = \frac{\kappa}{(VOLL-c)(1-\Psi(\tilde{x}_k + X_{K-1}(k)))} \\ &\geq \frac{\kappa}{(VOLL-c)(1-\Psi(x_k + X_{K-1}(k)))}. \end{aligned}$$

Consequently,

$$-(VOLL-c)\Psi^{'}(x_k + X_{K-1}(k))x_k + (VOLL-c)(1-\Psi(x_k + X_{K-1}(k))) > \kappa, \text{ and}$$

$$\frac{\partial E_\omega[\Pi^k(\omega, x_k, x_{-k})]}{\partial x_k} > 0.$$

Similarly, for $\tilde{x}_k < x_k < x_k^{max}$

$$1-\frac{\Psi'(x_k+X_{K-1}(k))}{(1-\Psi(x_k+X_{K-1}(k)))}x_k \quad < 1-\frac{\Psi'(\tilde{x}_k+X_{K-1}(k))}{(1-\Psi(\tilde{x}_k+X_{K-1}(k)))}\tilde{x}_k = \frac{\kappa}{(VOLL-c)(1-\Psi(\tilde{x}_k+X_{K-1}(k)))}$$
$$\leq \frac{\kappa}{(VOLL-c)(1-\Psi(x_k+X_{K-1}(k)))},$$

which implies with the same argument that

$$\frac{\partial E_\omega[\Pi^k(\omega,x_k,x_{-k})]}{\partial x_k} < 0.$$

We can conclude in this case that $E_\omega[\Pi^k(\omega,x_k,x_{-k})]$ is an increasing function for $x_k \in [x_k^{min},\tilde{x}_k]$ and a decreasing function for $x_k \in [\tilde{x}_k,x_k^{max}]$. Then we can conclude that $E_\omega[\Pi^k(\omega,x_k,x_{-k})]$ is strictly quasiconcave on $[x_k^{min},x_k^{max}]$ by using the above arguments.

Case (2) $x_k^{min}=0$ and $x_k^{max}>0$: $E_\omega[\Pi^k(\omega,x_k,x_{-k})]$ is continuous and decreasing on (x_k^{max},∞). Moreover, from (4.23) and Assumption (4.3.3) (ii), we know that

$$\frac{\partial E_\omega[\Pi^k(\omega,x_k^{min},x_{-k})]}{\partial x_k} = VOLL-c-\kappa > 0, \quad \frac{\partial E_\omega[\Pi^k(\omega,x_k^{max},x_{-k})]}{\partial x_k} < 0.$$

Then there exists an $\tilde{x}_k \in (0,x_k^{max})$ such that $0 \in \dfrac{\partial E_\omega[\Pi^k(\omega,\tilde{x}_k,x_{-k})]}{\partial x_k}$ and we are back in Case (1). Thus, $E_\omega[\Pi^k(\omega,x_k,x_{-k})]$ is a strictly quasiconcave function.

Case (3) $x_k^{max}=0$: Then $E_\omega[\beta_k^*(\omega,x_k,x_{-k})]=0$ and $E_\omega[\Pi^k(\omega,x_k,x_{-k})]=-\kappa x_k$ on $[0,\infty)$. Hence, $E_\omega[\Pi^k(\omega,x_k,x_{-k})]$ is decreasing and strictly quasiconcave.▌

Proof of Lemma 4.3.11. One can easily see that the strategy space is nonempty since $x_k=0$ is feasible for all firms. Moreover, there cannot be any optimal strategy $x_k^*>\overline{D}$ since $\dfrac{\partial E_\omega[\Pi^k(\omega,x_k,x_{-k})]}{\partial x_k}=-\kappa<0$ for all $x_k>\overline{D}$. ▌

Proof of Lemma 4.3.12. By Assumption 4.3.3 (ii), $\sum_{k=1}^{K} x_k^* \geq \underline{D}$. Let x^1 be the the vector of investment decisions of all firms where the total generation capacity is equal to the maximum demand level $\overline{D}$; that is $x^1 \in \{x_k^1, k\in 1,\dots K|\sum_{k=1}^K x_k^1 = \overline{D}\}$. Now assume any candidate x^2 for the equilibrium of all firms where $x^2 \in \{x_k^2, k\in 1,\dots K|\sum_{k=1}^K x_k^2 > \overline{D}\}$. Then the following holds:
(i) At both x^1 and x^2, the market price is equal to c for all demand realizations and the expected scarcity rents of all firms are zero which implies that all firms have negative profits equal to $-\kappa x_k^1$ and $-\kappa x_k^2$, respectively.
(ii) Since $x_k^1 < x_k^2$ for at least one firm k, the profit of at least one firm decreases from x^1 to x^2; that is,

$-\kappa x_k^2 < -\kappa x_k^1$ for at least one firm k.
Then x^2 cannot be an equilibrium since at least one firm has incentive to deviate from this point to x^1 to increase its profit. Thus $\sum_{k=1}^{K} x_k^* \leq \overline{D}$ holds at equilibrium. ▮

Proof of Theorem 4.3.13. From Lemma 4.3.11, we know that the strategy spaces of the firms are non-empty, compact and convex. Continuity and quasiconcavity of their payoff functions directly follow from Theorems 4.3.8 and 4.3.10, respectively. Then by utilizing Proposition 4.3.7, there exits a Nash equilibrium.

Each firm $k \in K$ is interested in finding a solution x_k^* which maximizes its expected profit for given x_{-k}:

$$x_k^* = \underset{x_k \geq 0}{\text{argmax}} \quad E_\omega[\Pi^k(\omega, x_k, x_{-k})]\}.$$

Let $\chi^k(x_{-k})$ denote the solution set of firm k for given x_{-k}. Then a Nash equilibrium is a point such that $x_k^* \in \chi^k(x_{-k}^*)$ for all $k \in K$ and the following first-stage optimality conditions will hold for all firms at an equilibrium of the two-stage game:

$$0 \leq -\frac{\partial E_\omega[\Pi^k(\omega, x_k^*, x_{-k}^*)]}{\partial x_k} \quad \perp \quad x_k^* \geq 0 \quad \forall k. \tag{4.25}$$

Next we show that *(i)* no asymmetric equilibria exist for the symmetric game and *(ii)* for symmetric equilibria, denoted by x^* for each firm, (4.8) holds and $\underline{D} \leq Kx^* \leq \overline{D}$, *(iii)* finally symmetric equilibrium satisfying (4.8) is unique.

(i) First we assume that there is an asymmetric equilibrium. Any candidate x^* for an asymmetric equilibrium of arbitrary firms i and j can be ordered as $0 \leq x_i^* < x_j^*$. Next we show by contradiction that an asymmetric equilibrium cannot exist since the first-stage optimality conditions of firms i and j are not simultaneously satisfied at (x_i^*, x_j^*):

- If $0 < x_i^* < x_j^*$, then the first-stage optimality conditions (4.25) for both firms i and j should satisfy the following:

$$\frac{\partial E_\omega[\Pi^i(\omega, x_i^*, x_{-i}^*)]}{\partial x_i} = \frac{\partial E_\omega[\Pi^j(\omega, x_j^*, x_{-j}^*)]}{\partial x_j} = 0.$$

We next show that whenever one of the equations above holds, the other cannot hold. Let

$$\frac{\partial E_\omega[\Pi^i(\omega, x_i^*, x_{-i}^*)]}{\partial x_i} = -(VOLL-c)\Psi'(\sum_{k=1}^{K} x_k^*)x_i^* + (VOLL-c)(1-\Psi(\sum_{k=1}^{K} x_k^*)) - \kappa = 0.$$

If the above equation holds then we get

$$\frac{\partial E_\omega[\Pi^j(\omega,x_j^*,x_{-j}^*)]}{\partial x_j} = -(VOLL-c)\Psi'(\sum_{k=1}^{K} x_k^*)x_j^* + (VOLL-c)(1-\Psi(\sum_{k=1}^{K} x_k^*)) - \kappa < 0 \text{ since } x_i^* < x_j^*.$$

- If $0 = x_i^* < x_j^*$, then the first-stage optimality conditions of firm i and j should satisfy

$$\frac{\partial E_\omega[\Pi^i(\omega,0,x_{-i}^*)]}{\partial x_i} = (VOLL-c)(1-\Psi(\sum_{k=1}^{K} x_k^*)) - \kappa \leq 0 \text{ and}$$

$$\frac{E_\omega[\Pi^j(\omega,x_j^*,x_{-j}^*)]}{\partial x_j} = 0.$$

By using a similar reasoning to the case above, whenever the first equation holds, then

$$\frac{\partial E_\omega[\Pi^k(\omega,x_j^*,x_{-j}^*)]}{\partial x_j} = -(VOLL-c)\Psi'(\sum_{k=1}^{K} x_k^*)x_j^* + (VOLL-c)(1-\Psi(\sum_{k=1}^{K} x_k^*)) - \kappa < 0 \text{ since } x_j^* > 0.$$

Hence, no asymmetric equilibria exits.

(ii) Any symmetric equilibrium can be denoted by $x^* = x_1^* = \ldots = x_K^*$. Next we identify the symmetric equilibrium by using Lemma 4.3.12 and the optimality conditions of the first stage:

- By Lemma 4.3.12, $\underline{D} \leq Kx^* \leq \overline{D}$. Thus, $\frac{\underline{D}}{K} \leq x^* \leq \frac{\overline{D}}{K}$.

- Since $\frac{\underline{D}}{K} \leq x^* \leq \frac{\overline{D}}{K}$, we are either in Case (1) or in Case (2) (i.e., $\underline{D} = 0$) of *Proof of Theorem 4.3.10*. Hence, there exists x^* which satisfies $\frac{\partial E_\omega[\Pi^k(\omega,x^*,(K-1)x^*)]}{\partial x_k} = 0$. By using the equality (4.24) given in *Proof of Theorem 4.3.10*, we get

$$x^* = \frac{(VOLL-c)(1-\Psi(Kx^*)) - \kappa}{(VOLL-c)\Psi'(Kx^*)}.$$

(iii) From the above arguments we know that each firm's first stage problem (4.26) is equivalent and has at least one symmetric equilibrium $x^* = x_1^* = \ldots = x_K^*$. Thus, at equilibrium, we know that x^* will be the optimal investment strategy of firm k when $x_{-k} := ex^*$ (where e is the vector of 1's of appropriate dimension). Therefore for given $x_{-k} := ex^*$, we would like to find x^* which is the solution to the following

problem of any firm k:

$$x^* = \underset{x_k \in S}{\operatorname{argmax}} E_\omega[\Pi(\omega, x_k, ex^*)], \tag{4.26}$$

By Lemma 4.3.12, $S := \{x_k | \underline{D} - (K-1)x^* \leq x_k \leq \overline{D} - (K-1)x^*\}$ which is equivalent to $x_k \in [x_k^{min}, x_k^{max}]$. By Theorem 4.3.10, we know that if the hazard rate function $H(s) = \Psi'(s)/(1-\Psi(s))$ is monotone increasing, then $E_\omega[\Pi(\omega, x_k, ex^*)]$ is strictly quasiconcave on S. Thus, there exists a unique x^* which maximizes the problem (4.26).∎

Proof of Lemma 4.3.15. From derivation of (4.20), recall that for given x_2 and realization ω,

$$\beta_1^*(\omega, x_1, x_2) = 0 I_{\{d(\omega) \leq x_1\}} + (c_2 - c_1) I_{\{x_1 < d(\omega) \leq x_1 + x_2\}} + (VOLL - c_1) I_{\{x_1 + x_2 < d(\omega)\}}$$

and hence:

$$\begin{aligned} E_\omega[\beta_1^*(\omega, x_1, x_2)] &= \int_{x_1}^{x_1+x_2} (c_2 - c_1) d(\Psi(s)) + \int_{x_1+x_2}^{d_{max}} (VOLL - c_1) d(\Psi(s)) \\ &= (c_2 - c_1)(\Psi(x_1 + x_2) - \Psi(x_1)) + (VOLL - c_1)(1 - \Psi(x_1 + x_2)) \\ &= VOLL - c_1 - (c_2 - c_1)\Psi(x_1) - (VOLL - c_2)\Psi(x_1 + x_2). \end{aligned}$$

When demand is uniformly distributed over $[d_{min}, d_{max}]$, $\Psi(s) = \dfrac{s - d_{min}}{d_{max} - d_{min}}$ is the cumulative distribution function. Therefore:

1) If $0 \leq x_1 < d_{min} - x_2$, then $\Psi(x_1) = \Psi(x_1 + x_2) = 0$ and $E_\omega[\beta_1^*(\omega, x_1, x_2)] = VOLL - c_1$.

2) If $d_{min} - x_2 \leq x_1 < d_{min}$, then $\Psi(x_1) = 0$. By plugging in $\Psi(x_1 + x_2) = \dfrac{x_1 + x_2 - d_{min}}{d_{max} - d_{min}}$, we get:

$$E_\omega[\beta_1^*(\omega, x_1, x_2)] = VOLL - c_1 - \frac{(VOLL - c_2)(x_2 - d_{min})}{d_{max} - d_{min}} - \frac{(VOLL - c_2)x_1}{d_{max} - d_{min}}.$$

It is easily seen that $E_\omega[\beta_1^*(\omega, x_1, x_2)]$ is a linear function where

$$VOLL - c_1 - \frac{(VOLL - c_2)(x_2 - d_{min})}{d_{max} - d_{min}} \text{ and } b_{\beta_1}^1 = \frac{(VOLL - c_2)}{d_{max} - d_{min}} > 0.$$

3) For $d_{min} \leq x_1 < d_{max} - x_2$, $\Psi(x_1) = \dfrac{x_1 - d_{min}}{d_{max} - d_{min}} \geq 0$ and $\Psi(x_1 + x_2) = \dfrac{x_1 + x_2 - d_{min}}{d_{max} - d_{min}} \geq 0$. Hence,

$$E_\omega[\beta_1^*(\omega, x_1, x_2)] = \frac{(VOLL - c_1)d_{max}}{d_{max} - d_{min}} - \frac{(VOLL - c_2)x_2}{d_{max} - d_{min}} - \frac{(VOLL - c_1)}{d_{max} - d_{min}} x_1.$$

$E_\omega[\beta_1^*(\omega,x_1,x_2)]$ is a linear function where

$$a_{\beta_1}^2 = \frac{(VOLL-c_1)d_{max}}{d_{max}-d_{min}} - \frac{(VOLL-c_2)x_2}{d_{max}-d_{min}} \text{ and } b_{\beta_1}^2 = \frac{(VOLL-c_1)}{d_{max}-d_{min}} > 0.$$

4) For $d_{max}-x_2 \leq x_1 < d_{max}$, $\Psi(x_1) = \dfrac{x_1-d_{min}}{d_{max}-d_{min}} \geq 0$, and $\Psi(x_1+x_2) = 1$. Then,

$$E_\omega[\beta_1^*(\omega,x_1,x_2)] = \frac{(c_2-c_1)d_{max}}{d_{max}-d_{min}} - \frac{(c_2-c_1)}{d_{max}-d_{min}}x_1,$$

which is again a linear function with $a_{\beta_1}^3 = \dfrac{(c_2-c_1)d_{max}}{d_{max}-d_{min}}$ and $b_{\beta_1}^3 = \dfrac{(c_2-c_1)}{d_{max}-d_{min}} > 0$.

5) If $x_1 \geq d_{max}$, then $\Psi(x_1) = \Psi(x_1+x_2) = 1$ and $E_\omega[\beta_1^*(\omega,x_1,x_2)] = 0$.
Similarly, for a given x_1 and realization ω,

$$\beta_2^*(\omega,x_2,x_1) = 0I_{\{d(\omega)\leq x_1+x_2\}} + (VOLL-c_2)I_{\{x_1+x_2<d(\omega)\}} \text{ which yields}$$

$$\begin{aligned} E_\omega[\beta_2^*(\omega,x_2,x_1)] &= \textstyle\int_{x_1+x_2}^{d_{max}}(VOLL-c_2)d(\Psi(s)) \\ &= (VOLL-c_2)(1-\Psi(x_1+x_2)) \\ &= VOLL-c_2-(VOLL-c_2)\Psi(x_1+x_2). \end{aligned}$$

When we have $d \sim U[d_{min}, d_{max}]$, we get:

1) If $0 \leq x_2 < d_{min}-x_1$, then $\Psi(x_1+x_2) = 0$ and $E_\omega[\beta_2^*(\omega,x_2,x_1)] = VOLL-c_2$.

2) For $d_{min}-x_1 \leq x_2 < d_{max}-x_1$,

$$E_\omega[\beta_2^*(\omega,x_2,x_1)] = \frac{(VOLL-c_2)d_{max}}{d_{max}-d_{min}} - \frac{(VOLL-c_2)x_1}{d_{max}-d_{min}} - \frac{(VOLL-c_2)}{d_{max}-d_{min}}x_2.$$

Clearly this is a linear function with

$$a_{\beta_2} = \quad \frac{(VOLL-c_2)d_{max}}{d_{max}-d_{min}} - \frac{(VOLL-c_2)x_1}{d_{max}-d_{min}} \text{ and } \beta_{\beta_2} = \quad \frac{(VOLL-c_2)}{d_{max}-d_{min}} > 0.$$

3) If $x_2 \geq d_{max}-x_1$, then $\Psi(x_1+x_2) = 1$ and $E_\omega[\beta_2^*(\omega,x_2,x_1)] = 0$.∎

Proof of Lemma 4.3.16. We can write the expected profit functions as $E_\omega[\Pi_1(\omega,x_1,x_2)] = (E_\omega[\beta_1^*(\omega,x_1,x_2)] - \kappa_1)x_1$ and $E_\omega[\Pi_2(\omega,x_2,x_1)] = (E_\omega[\beta_2^*(\omega,x_2,x_1)] - \kappa_2)x_2$. From Lemma 4.3.15, we know that both $E_\omega[\beta_1^*(\omega,x_1,x_2)]$ and $E_\omega[\beta_2^*(\omega,x_2,x_1)]$ are continuous piecewise linear functions of $x_1 \in \mathscr{R}_+$ and $x_2 \in \mathscr{R}_+$, respectively; therefore the expected profit functions are continuous as well.

They are also differentiable w.r.t. $x_1 \in \mathscr{R}_+$ and $x_2 \in \mathscr{R}_+$ in each region, respectively, due to the differentiability of expected scarcity rent functions in those regions. However, they are not differentiable at the breakpoints of the corresponding expected scarcity rent functions.

Given x_2, $E_\omega[\Pi_1(\omega,x_1,x_2)]$ is concave in x_1 in each region $0 \leq x_1 < d_{min} - x_2$, $d_{min} - x_2 \leq x_1 < d_{min}$, $d_{min} \leq x_1 < d_{max} - x_2$, $d_{max} - x_2 \leq x_1 < d_{max}$, and $x_1 \geq d_{max}$ since it is a linear or quadratic function in these regions. However, it is not necessarily quasiconcave on $\mathscr{R}_+$. On the other hand, given x_1, $E_\omega[\Pi_2(\omega,x_2,x_1)]$ is an increasing linear function of x_2 on $(0, d_{min} - x_1)$, concave on $[d_{min} - x_1, d_{max} - x_1]$, and a decreasing linear function on $(d_{max} - x_1, \infty)$. Hence it is a quasiconcave function on $\mathscr{R}_+$.▮

Proof of Proposition 4.4.3. From the illustration in Figure 4.3 and equation (4.13), we know that $\beta_k^*(\omega,x_k,x_{-k})$ has a positive value which is equal to $P(\omega,X_{K-1}(k)+x_k)-c$ when $\alpha(\omega) > \alpha(\widehat{\omega})$:

$$\begin{aligned}\beta_k^*(\omega,x_k,x_{-k}) = \quad & 0I\{\alpha_{min} \leq \alpha(\omega) \leq \alpha(\widehat{\omega})\} \\ + \quad & [\alpha(\omega) - \gamma\cdot(X_{K-1}(k)+x_k) - c]I\{\alpha(\widehat{\omega}) < \alpha(\omega) \leq \alpha_{max}\}.\end{aligned}$$

Then calculation of $E_\omega[\beta_k^*(\omega,x_k,x_{-k})]$ will yield the following:

$$\begin{aligned}E_\omega[\beta_k^*(\omega,x_k,x_{-k})] &= \int_{\alpha(\widehat{\omega})}^{\alpha_{max}} [s-\gamma\cdot(X_{K-1}(k)+x_k)-c]d(\Phi(s)) \\ &= \int_{\alpha(\widehat{\omega})}^{\alpha_{max}} sd(\Phi(s)) - [c+\gamma\cdot(X_{K-1}(k)+x_k)][1-\Phi(\alpha(\widehat{\omega})].\end{aligned}$$

By using integration by parts and the equality $\alpha(\widehat{\omega}) = c+\gamma\cdot(X_{K-1}(k)+x_k)$,

$$\begin{aligned}E_\omega[\beta_k^*(\omega,x_k,x_{-k})] &= s\Phi(s)|_{\alpha(\widehat{\omega})}^{\alpha_{max}} - \int_{\alpha(\widehat{\omega})}^{\alpha_{max}} \Phi(s)ds - \alpha(\widehat{\omega})[1-\Phi(\alpha(\widehat{\omega}))] \\ &= \alpha_{max}\Phi(\alpha_{max}) - \int_{\alpha(\widehat{\omega})}^{\alpha_{max}} \Phi(s)ds - \alpha(\widehat{\omega}).\end{aligned}$$

Since $\Phi(\alpha_{max})$=1, the above equation yields

$$\begin{aligned}E_\omega[\beta_k^*(\omega,x_k,x_{-k})] &= \alpha_{max} - \alpha(\widehat{\omega}) - \int_{\alpha(\widehat{\omega})}^{\alpha_{max}} \Phi(s)ds \\ &= \int_{\alpha(\widehat{\omega})}^{\alpha_{max}} (1-\Phi(s))ds.▮\end{aligned}$$

Proof of Lemma 4.4.4. Φ is a differentiable function and $\alpha(\widehat{\omega}) = c+\gamma\cdot(X_{K-1}(k)+x_k)$, which is the limit of the integral in (4.14), is also differentiable w.r.t. $x_k \in \mathscr{R}_+$. Then $E_\omega[\beta_k^*(\omega, x_k, x_{-k})]$ is differentiable w.r.t. $x_k \in \mathscr{R}_+$. By using Leibniz rule and the fundamental theorem of calculus for differentiation of integral, we get:

$$\begin{aligned}\frac{\partial E_\omega[\beta_k^*(\omega, x_k, x_{-k})]}{\partial x_k} &= -\frac{\partial \alpha(\widehat{\omega})}{\partial x_k}(1-\Phi(\alpha(\widehat{\omega}))) \\ &= -\gamma \cdot (1-\Phi(\alpha(\widehat{\omega}))) \leq 0.\end{aligned}$$

Moreover, the second derivative of $E_\omega[\beta_k^*(\omega, x_k, x_{-k})]$ w.r.t. x_k will yield the following:

$$\frac{\partial^2 E_\omega[\beta_k^*(\omega, x_k, x_{-k})]}{\partial x_k^2} = \gamma^2 \Phi'(\alpha(\widehat{\omega})) \geq 0.$$

Hence, $E_\omega[\beta_k^*(\omega, x_k, x_{-k})]$ is a non-increasing convex function of $x_k \in \mathscr{R}_+$. ▮

Proof of Theorem 4.4.5. Since $E_\omega[\Pi^k(\omega, x_k, x_{-k})] = (E_\omega[\beta_k^*(\omega, x_k, x_{-k})] - \kappa_k)x_k$, *(i)* immediately follows from Lemma 4.4.4.

For *(ii)*, note that

$$\log(E_\omega[\Pi^k(\omega, x_k, x_{-k})]) = \log x_k + \log(E_\omega[\beta_k^*(\omega, x_k, x_{-k})] - \kappa).$$

We know that $\log x_k$ is strictly concave. We next show that if the probability density function Φ' is logconcave then $B(x_k) := E_\omega[\beta_k^*(\omega, x_k, x_{-k})] - \kappa$ is logconcave.

By using Theorem 3 of Bagnoli and Bergstrom (2005), we know that log-concavity of Φ' implies the log-concavity of the right hand integral of the reliability function defined by

$$H(t) = \int_t^{\alpha_{max}} (1-\Phi(s))ds \quad \text{for} \quad t \in (\alpha_{min}, \alpha_{max}),$$

which implies

$$\frac{\partial^2(logH(t))}{\partial t^2} = \frac{\Phi'(t)\int_t^{\alpha_{max}}(1-\Phi(s))ds - (1-\Phi(t))^2}{(H(t))^2} \leq 0$$

and

$$\overline{H}(t) := \Phi'(t)\int_t^{\alpha_{max}}(1-\Phi(s))ds - (1-\Phi(t))^2 \leq 0.$$

By using the above inequality we can prove that $\dfrac{\partial^2 \log(B(x_k))}{\partial x_k^2} \leq 0$:

$$\begin{aligned}
\frac{\partial^2 \log(B(x_k))}{\partial x_k^2} = \quad & \gamma^2 \frac{\Phi'(\alpha(\widehat{\omega})) \int_{\alpha(\widehat{\omega})}^{\alpha_{max}} (1-\Phi(s))ds - (1-\Phi(\alpha(\widehat{\omega})))^2}{(B(x_k))^2} \\
& -\gamma^2 \frac{\kappa \Phi'(\alpha(\widehat{\omega}))}{(B(x_k))^2} \\
& = \gamma^2 \frac{\overline{H}(t)|_{t=\alpha(\widehat{\omega})}}{(B(x_k))^2} - \gamma^2 \frac{\kappa \Phi'(\alpha(\widehat{\omega}))}{(B(x_k))^2} \leq 0.
\end{aligned}$$

Hence, $(E_\omega[\beta_k^*(\omega, x_k, x_{-k})] - \kappa)$ is a logconcave function of x_k. The sum of a strictly concave function, $\log x_k$, and a concave function, $\log(E_\omega[\beta_k^*(\omega, x_k, x_{-k})] - \kappa)$, is strictly concave. Thus, $E_\omega[\Pi^k(\omega, x_k, x_{-k})]$ is strictly logconcave in x_k. ▮

Proof of Lemma 4.4.6. The strategy space is nonempty since the strategy $x_k = 0$ is feasible for all firms. For any firm k, if $x_k > \dfrac{\alpha_{max} - c}{\gamma}$ then, by Proposition 4.4.3, $E_\omega[\beta_k^*(\omega, x_k, x_{-k})] = 0$ (note that $\alpha(\widehat{\omega}) > \alpha_{max}$ for all $x_k > \dfrac{\alpha_{max} - c}{\gamma}$). Then for $x_k > \dfrac{\alpha_{max} - c}{\gamma}$, $\dfrac{\partial E_\omega[\Pi^k(\omega, x_k, x_{-k})]}{\partial x_k} = -\kappa < 0$ which implies that firm k's profit for this set of investment strategies is lower than its profit gained by the strategies in $S_k := [0, \dfrac{\alpha_{max} - c}{\gamma}]$ and therefore can be excluded from its strategy space.▮

Proof of Theorem 4.4.7. From Lemma 4.4.6, we know that the strategy spaces are non-empty, compact, and convex. Continuity of the payoff functions directly follows from Theorem 4.4.5. Boyd and Vandenberghe (2004) gives a composition theorem which shows the preservation of quasiconcavity under monotonic functions (see Section 3.4.4 on pages 101/102 and Section 3.5 on page 104). Utilizing this result, we know that (strict) log-concavity implies (strict) quasiconcavity of firms' payoff functions. A similar result can also be found in Theorem 3.3 of Avriel (1972) under the notion of ρ-concavity. Then by utilizing Theorem 4.4.5 and Proposition 4.3.7, there exists a Nash equilibrium.

$E_\omega[\Pi^k(\omega, x_k, x_{-k})]$ is differentiable and strictly logconcave in x_k. Each firm $k \in K$ is interested in finding a solution x_k^* which maximizes its expected profit for given x_{-k}:

$$x_k^* = \underset{x_k \geq 0}{\operatorname{argmax}} \quad E_\omega[\Pi^k(\omega, x_k, x_{-k})]\}.$$

Let $\chi^k(x_{-k})$ denote the solution set of firm k for given x_{-k}. Then a Nash equilibrium is a point such that $x_k^* \in \chi^k(x_{-k}^*)$ for all $k \in K$ and the following first-stage optimality conditions will hold for all firms at an equilibrium of the two-stage game:

$$0 \leq -\frac{\partial E_\omega[\Pi^k(\omega, x_k^*, x_{-k}^*)]}{\partial x_k} \quad \perp \quad x_k^* \geq 0, \forall k \in K, \tag{4.27}$$

where
$\frac{\partial E_\omega[\Pi^k(\omega,x_k^*,x_{-k}^*)]}{\partial x_k} = \int_{\alpha(\widehat{\omega})}^{\alpha_{max}} (1-\Phi(s))ds - \gamma\cdot(1-\Phi(\alpha(\widehat{\omega})))x_k^* - \kappa$ and $\alpha(\widehat{\omega}) = c + \gamma\sum_{j=1}^{K} x_j^*$.

Next we show that (i) no asymmetric equilibria exist, (ii) for symmetric equilibria, denoted by x^* for each firm, (4.16) holds and $0 < x^* < \frac{\alpha_{max}-c}{K\gamma}$, and (iii) finally symmetric equilibrium satisfying (4.16) is unique.

(i) We first assume that there is an asymmetric equilibrium. Any candidate x^* for an asymmetric equilibrium of arbitrary firms i and j can be ordered as $0 \le x_i^* < x_j^*$. Next we show by contradiction that an asymmetric equilibrium cannot exist since the first stage optimality conditions given in (4.27) are not simultaneously satisfied for firms i and j .

- If $0 < x_i^* < x_j^*$, then the first stage optimality conditions (4.27) for firms i and j should satisfy the following

$$\frac{\partial E_\omega[\Pi^i(\omega,x_i^*,x_{-i}^*)]}{\partial x_i} = \frac{\partial E_\omega[\Pi^j(\omega,x_j^*,x_{-j}^*)]}{\partial x_j} = 0.$$

  We next show that whenever one of the equations above holds, the other cannot hold. Let

$$\frac{\partial E_\omega[\Pi^i(\omega,x_i^*,x_{-i}^*)]}{\partial x_i} = \int_{\alpha(\widehat{\omega})}^{\alpha_{max}} (1-\Phi(s))ds - \gamma\cdot(1-\Phi(\alpha(\widehat{\omega})))x_i^* - \kappa = 0.$$

  If the above equation holds then since $x_i^* < x_j^*$,

$$\frac{\partial E_\omega[\Pi^j(\omega,x_j^*,x_{-j}^*)]}{\partial x_j} = \int_{\alpha(\widehat{\omega})}^{\alpha_{max}} (1-\Phi(s))ds - \gamma\cdot(1-\Phi(\alpha(\widehat{\omega})))x_j^* - \kappa < 0.$$

- If $0 = x_i^* < x_j^*$, then the first-stage optimality conditions of firm i and j should satisfy

$$\frac{\partial E_\omega[\Pi^i(\omega,x_i^*,x_{-i}^*)]}{\partial x_i} = \int_{\alpha(\widehat{\omega})}^{\alpha_{max}} (1-\Phi(s))ds - \kappa \le 0, \text{ and}$$

$$\frac{\partial E_\omega[\Pi^i(\omega,x_j^*,x_{-i}^*)]}{\partial x_i} = 0.$$

  By using a similar reasoning to the case above, whenever the first equation holds then since $x_j^* > 0$

$$\frac{\partial E_\omega[\Pi^j(\omega,x_j^*,x_{-j}^*)]}{\partial x_j} = \int_{\alpha(\widehat{\omega})}^{\alpha_{max}} (1-\Phi(s))ds - \gamma\cdot(1-\Phi(\alpha(\widehat{\omega})))x_j^* - \kappa < 0.$$

Hence, no asymmetric equilibria exits.

(ii) Any symmetric equilibrium can be denoted by $x^* = x_1^* = \ldots = x_K^*$. Next we identify the symmetric equilibrium by using Assumption 4.4.2 and the optimality conditions of the first stage:

– By Assumption 4.4.2, $x^* > 0$.

– At $x^* > 0$, the following optimality conditions are satisfied for each firm

$$\frac{\partial E_\omega[\Pi^k(\omega, x^*, (K-1)x^*]}{\partial x_k} = \int_{\alpha(\widehat{\omega})}^{\alpha_{max}} (1 - \Phi(s))ds - \gamma \cdot (1 - \Phi(\alpha(\widehat{\omega})))x^* - \kappa = 0.$$

Hence, $\int_{\alpha(\widehat{\omega})}^{\alpha_{max}} (1 - \Phi(s))ds - \kappa = \gamma \cdot (1 - \Phi(\alpha(\widehat{\omega})))x^*$ which yields (4.16).

(iii) From the above arguments we know that each firm's first stage problem (4.28) is equivalent and has at least one symmetric equilibrium $x^* = x_1^* = \ldots = x_K^*$. Thus, at equilibrium, we know that x^* will be the optimal investment strategy of firm k when $x_{-k}^* := ex^*$ (where e is the vector of 1's of appropriate dimension). Therefore for given $x_{-k} := ex^*$, we would like to find x^* which is the solution to the following problem of firm k:

$$x^* = \underset{x_k \geq 0}{\operatorname{argmax}} \, E_\omega[\Pi(\omega, x_k, ex^*)], \tag{4.28}$$

By Theorem 4.4.5 (ii), we know that if Φ' is logconcave, then $E_\omega[\Pi(\omega, x_k, ex^*)]$ is strictly logconcave for $x_k \in \Re_+$ which implies its strict quasiconcavity for $x_k \in \Re_+$. Thus, there exists a unique x^* maximizing $E_\omega[\Pi(\omega, x_k, ex^*)]$ in (4.28).∎

Proof of Proposition 4.4.8. By using explicit formulation of linear price demand curve, we get:

$$\begin{aligned} E_\omega[\beta_k^*(\omega, x_k, x_{-k})] = \; & \sum_{m=k}^{K-1} \int_{\alpha(\overline{\omega}_m)}^{\alpha(\underline{\omega}_{m+1})} (s - c_k - \gamma \cdot (X_{m-1}(k) + x_k))\Phi'(s)ds \\ & + \int_{\alpha(\overline{\omega}_K)}^{\alpha_{max}} (s - c_k - \gamma \cdot (X_{K-1}(k) + x_k))\Phi'(s)ds \\ & + \sum_{m=k+1}^{K} \int_{\alpha(\underline{\omega}_m)}^{\alpha(\overline{\omega}_m)} (c_m - c_k)\Phi'(s)ds. \end{aligned}$$

Next we simplify the above formulation:

$$
\begin{aligned}
E_\omega[\beta_k^*(\omega,x_k,x_{-k})] = \; & \sum_{m=k}^{K-1}(\alpha(\underline{\omega}_{m+1})\Phi(\alpha(\underline{\omega}_{m+1})) - \alpha(\overline{\omega}_m)\Phi(\alpha(\overline{\omega}_m)) - \int_{\alpha(\overline{\omega}_m)}^{\alpha(\underline{\omega}_{m+1})}\Phi(s)ds \\
& - \sum_{m=k}^{K-1}(c_k+\gamma\cdot(X_{m-1}(k)+x_k))[\Phi(\alpha(\underline{\omega}_{m+1}))-\Phi(\alpha(\overline{\omega}_m))] \\
& + \sum_{m=k+1}^{K}(c_m-c_k)[\Phi(\alpha(\overline{\omega}_m))-\Phi(\alpha(\underline{\omega}_m))] \\
& +(\alpha_{max}\Phi(\alpha_{max})-\alpha(\overline{\omega}_K)\Phi(\alpha(\overline{\omega}_K))-\int_{\alpha(\overline{\omega}_K)}^{\alpha_{max}}\Phi(s)ds \\
& -(c_k+\gamma\cdot(X_{K-1}(k)+x_k))[\Phi(\alpha_{max})-\Phi(\alpha(\overline{\omega}_K)),]
\end{aligned}
$$

$$
\begin{aligned}
E_\omega[\beta_k^*(\omega,x_k,x_{-k})] \quad & = \sum_{m=k}^{K-1}(c_{m+1}-c_k)\Phi(\alpha(\underline{\omega}_{m+1})) - \sum_{m=k}^{K-1}(c_m-c_k)\Phi(\alpha(\overline{\omega}_m)) \\
& + \sum_{m=k+1}^{K}(c_m-c_k)\Phi(\alpha(\overline{\omega}_m)) - \sum_{m=k+1}^{K}(c_m-c_k)\Phi(\alpha(\underline{\omega}_m)) \\
& - \sum_{m=k}^{K-1}\int_{\alpha(\overline{\omega}_m)}^{\alpha(\underline{\omega}_{m+1})}\Phi(s)ds - \int_{\alpha(\overline{\omega}_K)}^{\alpha_{max}}\Phi(s)ds \\
& -(c_K-c_k)\Phi(\alpha(\overline{\omega}_K)) + \alpha_{max} - (c_k+\gamma\cdot(X_{K-1}(k)+x_k) \\
E_\omega[\beta_k^*(\omega,x_k,x_{-k})] \quad & = (c_K-c_k)\Phi(\alpha(\overline{\omega}_K)) - (c_K-c_k)\Phi(\alpha(\overline{\omega}_K)) + \alpha_{max} - (c_k+\gamma\cdot(X_{K-1}(k)+x_k)) \\
& - \sum_{m=k}^{K-1}\int_{\alpha(\overline{\omega}_m)}^{\alpha(\underline{\omega}_{m+1})}\Phi(s)ds - \int_{\alpha(\overline{\omega}_K)}^{\alpha_{max}}\Phi(s)ds, \\
E_\omega[\beta_k^*(\omega,x_k,x_{-k})] \quad & = \alpha_{max} - (c_k+\gamma\cdot(X_{K-1}(k)+x_k)) - \sum_{m=k}^{K-1}\int_{\alpha(\overline{\omega}_m)}^{\alpha(\underline{\omega}_{m+1})}\Phi(s)ds - \int_{\alpha(\overline{\omega}_K)}^{\alpha_{max}}\Phi(s)ds,
\end{aligned}
$$

Finally, we add $c_K - c_K$ to the equation above, then we get:

$$
E_\omega[\beta_k^*(\omega,x_k,x_{-k})] \quad = (c_K-c_k) + \int_{\alpha(\overline{\omega}_K)}^{\alpha_{max}}(1-\Phi(s))ds - \sum_{m=k}^{K-1}\int_{\alpha(\overline{\omega}_m)}^{\alpha(\underline{\omega}_{m+1})}\Phi(s)ds. \blacksquare
$$

Proof of Lemma 4.4.9. $P(\omega,\cdot)$ and Φ are differentiable w.r.t. their arguments and Φ is independent from x_k. From (4.19) we know that, for $m \geq k$, $(\alpha(\overline{\omega}_m), \alpha(\underline{\omega}_{m+1}))$ are differentiable w.r.t. $x_k \in \mathscr{R}_+$. Then being a sum of differentiable functions in (4.17), $E_\omega[\beta_k^*(\omega, x_k, x_{-k})]$ is differentiable w.r.t. $x_k \in \mathscr{R}_+$. By using Leibniz rule and the fundamental theorem of calculus for differentiation of integral in (4.17) and by utilizing (4.19) which simplifies $P(\underline{\omega}_{m+1}, X_{m-1}(k)+x_k) = c_{m+1}$ and $P(\overline{\omega}_m, X_{m-1}(k)+x_k) = c_m$, we get

$$\begin{aligned}\frac{\partial E_\omega[\beta_k^*(\omega,x_k,x_{-k})]}{\partial x_k} = \quad & \sum_{m=k}^{K-1}[\frac{\partial \alpha(\underline{\omega}_{m+1})}{\partial x_k}(c_{m+1}-c_k)\Phi'(\alpha(\underline{\omega}_{m+1})) - \frac{\partial \alpha(\overline{\omega}_m)}{\partial x_k}(c_m - c_k)\Phi'(\alpha(\overline{\omega}_m))] \\ & + \sum_{m=k}^{K-1}\int_{\alpha(\overline{\omega}_m)}^{\alpha(\underline{\omega}_{m+1})} \frac{\partial P(s, X_{m-1}(k)+x_k)}{\partial x_k}\Phi'(s)ds \\ & - \frac{\partial \alpha(\overline{\omega}_K)}{\partial x_k}(c_K - c_k)\Phi'(\alpha(\overline{\omega}_K)) + \int_{\alpha(\overline{\omega}_K)}^{\alpha_{max}} \frac{\partial P(s, X_{K-1}(k)+x_k)}{\partial x_k}\Phi'(s)ds \\ & + \sum_{m=k+1}^{K}[\frac{\partial \alpha(\overline{\omega}_m)}{\partial x_k}(c_m - c_k)\Phi'(\alpha(\overline{\omega}_m)) - \frac{\partial \alpha(\underline{\omega}_m)}{\partial x_k}(c_m - c_k)\Phi'(\alpha(\underline{\omega}_m))].\end{aligned} \tag{4.29}$$

The expression in (4.29) can be simplified to:

$$\begin{aligned}\frac{\partial E_\omega[\beta_k^*(\omega,x_k,x_{-k})]}{\partial x_k} = \quad & \sum_{m=k}^{K-1}\int_{\alpha(\overline{\omega}_m)}^{\alpha(\underline{\omega}_{m+1})} \frac{\partial P(s, X_{m-1}(k)+x_k)}{\partial x_k}\Phi'(s)ds \\ & + \int_{\alpha(\overline{\omega}_K)}^{\alpha_{max}} \frac{\partial P(s, X_{K-1}(k)+x_k)}{\partial x_k}\Phi'(s)ds \\ & = -\gamma\sum_{m=k}^{K-1}[\Phi(\alpha(\underline{\omega}_{m+1})) - \Phi(\alpha(\overline{\omega}_m)] - \gamma[1-\Phi(\alpha(\overline{\omega}_K))] \\ & < 0.\end{aligned}$$

Thus, $E_\omega[\beta_k^*(\omega, x_k, x_{-k})]$ is decreasing.∎

Proof of Theorem 4.4.10. Since $E_\omega[\Pi^k(\omega, x_k, x_{-k})] = (E_\omega[\beta_k^*(\omega, x_k, x_{-k})] - \kappa_k)x_k$, *(i)* follows immediately from Lemma 4.4.9. For *(ii)*, we know that

$$E_\omega[\Pi^k(\omega, x_k, x_{-k})] = x_k(E_\omega[\beta_k^*(\omega, x_k, x_{-k})] - \kappa).$$

$E_\omega[\Pi^k(\omega, x_k, x_{-k})]$ is a differentiable function of x_k with

$$\frac{\partial E_\omega[\Pi^k(\omega, x_k, x_{-k})]}{\partial x_k} = x_k \cdot \frac{\partial E_\omega[\beta_k^*(\omega, x_k, x_{-k})]}{\partial x_k} + E_\omega[\beta_k^*(\omega, x_k, x_{-k})] - \kappa.$$

$\dfrac{\partial E_\omega[\Pi^k(\omega, x_k, x_{-k})]}{\partial x_k}$ can be explicitly formulated by using (4.18):

$$\begin{aligned}\frac{\partial E_\omega[\Pi^k(\omega, x_k, x_{-k})]}{\partial x_k} &= -x_k \cdot \gamma \sum_{m=k}^{K-1} \int_{\alpha(\overline{\omega}_m)}^{\alpha(\underline{\omega}_{m+1})} \Phi'(s)ds - x_k \cdot \gamma[1 - \Phi(\alpha(\overline{\omega}_K))] \\ &\quad + (c_K - c_k) + \int_{\alpha(\overline{\omega}_K)}^{\alpha_{max}} (1 - \Phi(s))ds - \sum_{m=k}^{K-1} \int_{\alpha(\overline{\omega}_m)}^{\alpha(\underline{\omega}_{m+1})} \Phi(s)ds, -\kappa.\end{aligned}$$

which can further be rewritten as

$$\begin{aligned}\frac{\partial E_\omega[\Pi^k(\omega, x_k, x_{-k})]}{\partial x_k} &= c_K - c_k - \kappa + \underbrace{\int_{\alpha(\overline{\omega}_K)}^{\alpha_{max}} [(1 - \Phi(s)) - x_k \cdot \gamma\Phi'(s)]ds}_{A_1(x_k)} \\ &\quad \underbrace{- \sum_{m=k}^{K-1} \int_{\alpha(\overline{\omega}_m)}^{\alpha(\underline{\omega}_{m+1})} (\Phi(s) + x_k \cdot \gamma\Phi'(s))ds}_{A_2(x_k)}.\end{aligned}$$

We know that $c_K - c_k - \kappa$ is constant. In addition under the condition $-\Phi'(s+C) - s \cdot \gamma\Phi''(s+C)(<) \leq 0$ for all $s, C \geq 0$, $A_1(x_k)$ and $A_2(x_k)$ are (decreasing) nonincreasing functions since $\dfrac{\partial A_1(x_k)}{\partial x_k} \leq 0$ and $\dfrac{\partial A_2(x_k)}{\partial x_k}(<) \leq 0$, which implies $\dfrac{\partial^2 E_\omega[\Pi^k(\omega, x_k, x_{-k})]}{\partial x_k^2}(<) \leq 0$. Hence, $E_\omega[\Pi^k(\omega, x_k, x_{-k})]$ is (strictly) concave if $-\Phi'(s+C) - s \cdot \gamma\Phi''(s+C)(<) \leq 0$ for all $s, C \geq 0$.∎

Proof of Lemma 4.4.11. The strategy space is nonempty since the strategy $x_k = 0$ is feasible for all firms. For any firm $k \in K$, if $x_k > \dfrac{\alpha_{max} - c_k}{\gamma}$ then $p_k^*(\omega, x_k, x_{-k}) = c_k$ for all $\omega \in \Omega$ when firm k's generation is positive. Thus, $E_\omega[\beta_k^*(\omega, x_k, x_{-k})] = 0$ for all $x_k > \dfrac{\alpha_{max} - c_k}{\gamma}$. Then for $x_k > \dfrac{\alpha_{max} - c_k}{\gamma}$, $\dfrac{\partial E_\omega[\Pi^k(\omega, x_k, x_{-k})]}{\partial x_k} = -\kappa < 0$ which implies that firm k's profit for this set of investment strategies is lower than its profit gained by the strategies in $S_k := [0, \dfrac{\alpha_{max} - c_k}{\gamma}]$ and therefore can be excluded from its strategy space. ∎

Proof of Theorem 4.4.12. From Lemma 4.4.11, we know that the strategy spaces are non-empty, compact, and convex. Continuity of the payoff functions directly follows from Theorem 4.4.10. Then by Proposition 4.3.7, there exits a Nash equilibrium. Furthermore, Rosen (1965) shows that there exists a unique equilibrium point when the payoff function of every player is strictly concave. By using the result of Rosen (1965), there exists a unique equilibrium since the expected profit function of each firm is strictly concave under the given condition. ∎

Chapter 5

Optimal Threshold Levels in Stochastic Fluid Models via Simulation-Based Optimization

Abstract

A number of important problems in production and inventory control involve optimization of multiple threshold levels or hedging points. We address the problem of finding such levels in a stochastic system whose dynamics can be modelled using generalized semi-Markov processes (GSMP). The GSMP framework enables us to compute several performance measures and their sensitivities from a single simulation run for a general system with several states and fairly general state transitions. We then use a simulation-based optimization method, sample-path optimization, for finding optimal hedging points. We report numerical results for systems with more than twenty hedging points and service-level type probabilistic constraints. In these numerical studies, our method performs quite well even on problems with nonconvex objective functions and probabilistic constraints, which are considered very difficult to solve by current standards. In addition, we obtain insights on the structure of optimal hedging points for much larger systems than is usually reported in the literature. In particular, we observe that production and demand variability has an important effect on the optimal threshold levels of a system. As the production and demand variability increases, the optimal threshold levels also increase to hold enough safety stock against the uncertainty in production and demand rates. Furthermore, as the utilization of a system converges to one, the optimal threshold levels get close to each other and converge to a single threshold level. Some applications falling into this framework include designing manufacturing flow controllers, using capacity options and subcontracting strategies, and coordinating production and marketing activities under demand uncertainty.

5.1 Introduction

In an influential paper, Kimemia and Gershwin (1983) propose a framework for studying production control problems as optimal control problems for stochastic fluid systems. This framework, usually referred to as manufacturing flow control, assumes that the production/inventory processes of interest are governed by flows that correspond to deterministic production and demand rates and that the randomness in the system can be captured by changes in production or demand rates according to some stationary stochastic process governing the possible input and output rates. Since then, there has been a strong interest in the development of models and in their analysis within this framework; Sethi et al. (2002) provide a comprehensive review of this literature.

The particular manufacturing flow control problem that we consider in this chapter is that of a single resource (a "machine" for simplicity) coping with a flow of demand whose rate varies randomly. The production capacity of the machine can take any of a number of alternative values according to some external stationary stochastic process. Similarly the demand rate at any given time is governed by another external and uncontrollable stationary stochastic process. The objective then is to dynamically select the production rate of the machine (within the capacity constraints) in order to minimize expected inventory and backorder costs. Under the general assumptions we consider here (multiple machine and demand states and general transition times), the exact solution of the optimal control problem is unknown and even if it were known the exact optimal policy is likely to be extremely complicated. On the other hand, there is a plausible class of policies called "hedging point" policies that are described by a few parameters and therefore are extremely useful for implementation in practice. A hedging point policy, in general, requires determining a parameter corresponding to each machine and demand state. The corresponding parameter (hedging point) acts as an inventory target threshold in each machine/demand state. The machine should produce at the maximum rate if the inventory level is below the current threshold, should stop if the inventory level is above the current threshold, and should produce at the rate that would keep the inventory level at the current threshold level when it reaches there.

As described further in Section 5.2, hedging point policies are attractive not only due to their simplicity but also because they turn out to be optimal in a number of important special cases studied in the literature. In addition, if the production policy is required to depend only on the inventory level (and not on elapsed times since previous transitions), hedging point policies seem to be the only practical alternative for implementation. In turn, these type of policies are used not only for designing manufacturing controllers, but also for using capacity options and subcontracting strategies, and for coordinating production and marketing decisions.

Focusing on the class of hedging point policies reduces the optimal control problem to the optimization of a finite number of parameters: the hedging points. This optimization problem is the main focus of this chapter. In particular, we propose a simulation-based approach, namely sample-path optimization, to tackle this problem. To this end, we first model the evolution of the system using a generalized semi-

Markov Process (GSMP) framework. Although in the past the GSMPs were mainly used for modeling systems with discrete entities, they can also be well suited for modeling fluid systems (see Suri and Fu (1994) and Gürkan (2000) for example). Here, we also adapt the GSMP framework to our continuous-time continuous-state space problem. Utilizing the GSMP representation, we then obtain gradient estimates of the performance measures using infinitesimal perturbation analysis (IPA) (see Ho and Cao (1991) and Glassermann (1991)). Finally, we implement the relatively recent technique of sample-path optimization to find the optimal hedging points.

Sample-path optimization is a simulation-based optimization method to optimize performance functions of complex stochastic systems; it can be used for providing solutions to difficult stochastic optimization problems (including ones with stochastic constraints), stochastic variational inequalities, and equilibrium models. The basic idea is to observe a fixed sample path (by using the method of common random numbers from the simulation literature), solve the resulting deterministic problem using fast and effective methods from nonlinear programming, and then use the resulting solutions to infer information about the solution of the original stochastic problem. In Section 5.2, we briefly explain the underlying basic ideas and give references in which more details can be found.

To date, there have been several successful implementations of simulation-based optimization in the context of manufacturing flow control. Most of those papers, mentioned in Section 5.2, use IPA coupled with some form of stochastic approximation (SA) technique on the optimization side. More precisely, they apply SA-IPA framework to particular models and/or establish the validity of that approach for those particular models. There seems to be a number of valid reasons for focusing on particular models. To start with, proving the validity and unbiasedness of the IPA estimate can be extremely challenging and is usually dependent on the particular model representation. Moreover, the practical implementation of stochastic approximation (or its variants) has to be carefully adapted and fine tuned to the context even when unbiased derivative estimates are available. Partly as a consequence of these, the existing literature doesn't go beyond the unconstrained optimization of two hedging points.

In contrast with most of the existing literature, we start by a fairly general single-stage model allowing any number of machine and demand states and general state transitions. Our main contributions are two-fold. First, using the GSMP framework, we obtain a flexible and general representation of the system evolution, and derive gradient estimates. Even though these estimates may be obtained by other means, the GSMP approach significantly facilitates obtaining transparent estimates by simulation and proving the unbiasedness of these estimates. Second, using sample-path optimization we avoid certain drawbacks associated with stochastic approximation type methods such as the number of parameters to fine tune, slow convergence, and ad-hoc handling of constraints. These two complementary features enable us to investigate much larger systems than is usually reported in comparable simulation-based optimization studies and to obtain insights on the structure of optimal hedging points for large systems. Furthermore, we also report numerical experiments with service-level type probabilistic constraints. Our highlights from these numerical experiments can be summarized as follows:

- In one of the systems, we ignore the demand variability and use the average demand rate instead. When we compare the solution of the system based on the average demand rate with the solution of the same system including demand variability, we observe significant differences between the optimal hedging points. Thus, the variability of demand has an important effect on the optimal hedging points. Therefore, considering only the average demand without taking into account its variability leads to incorrect decisions.

- The load (utilization) of a system depends on the average production and demand rates. In our numerical experiments, we observe that as the utilization of a system converges to one, the optimal hedging points get close to each other and converge to a single hedging point.

- The differences between the optimal hedging points of a system also depend on the transition diagram and the transition probabilities. The more the machine states communicate with each other and the more the state transition probabilities are close to uniform distribution, the more the optimal hedging points get close to each other. As the transition probabilities between the machine states become less uniform, the optimal hedging points move away from each other.

- In a system, as the production and demand variability increases, the optimal hedging points also increase to hold enough safety stock against the uncertainty in production and demand rates.

- In a system, as the ratio of holding and backorder costs gets smaller, that is the system has very low holding cost compared to its backorder cost, the average system cost gets flatter near the optimum points suggesting multiple optima.

- As also mentioned in Brémaud et al. (1997), the long run average cost function is generally not convex in systems with two or more hedging points, which we also observe in some of our numerical experiments. In some of the systems, we reach to different optimal hedging levels when we start from different initial points, which can be expected when dealing with nonconvex functions. However, we do not encounter any numerical difficulties in finding the local optimal solutions even in large systems despite the nonconvex nature of the average cost function and the probabilistic constraints. Furthermore, it is worth to note that these different local optimal solutions result in very similar performance measures. However, additional testing is necessary to be sure whether this multiplicity is due to several local isolated optima or multiple optima or both.

We would like to emphasize that although the main focus of this chapter is the analysis of a particular manufacturing flow control model, the scope of application for the developed techniques is potentially much wider. Threshold-based control of stochastic fluid models have useful applications in other related problems in manufacturing control (e.g., Tan and Gershwin (2004), Hu et al. (2004), Panayiotou and Cassandras (2006), Gershwin et al. (2009), Zhao and Melamed (2006, 2007), Melamed et al. (2010)) and queueing-based admission control and routing models commonly used in telecommunications (e.g., Sun et al. (2004), Wardi et al. (2010)). The general approach proposed here may be utilized with appropriate modifications for such models as well. Cassandras (2007) provides an overview of the theory of stochastic fluid models and their applications with threshold-based controllers in various fields such as finance, telecommunications, and engineering.

The rest of the chapter is organized as follows. In Section 5.2 we present a review of the related literature. Section 5.3 provides a detailed description of the problem. The GSMP framework and gradient estimation are presented in Section 5.4. The numerical results can be found in Section 5.5 and the conclusions are presented in Section 5.6. To improve readability, we deal with the details of our technical analysis in several appendices. Appendix 5.A provides a proof of a sufficient condition for the stability of the system under consideration. Appendix 5.B deals with so-called "similarity properties" needed for rigorous analysis of our gradient estimation. Appendix 5.C contains the proofs of several technical results of Section 5.4. Finally, Appendix 5.D gives the pseudo-code for simulation and gradient estimation.

5.2 Literature

The research in manufacturing flow control can roughly be classified in two directions: papers directly addressing optimal control issues and papers that analyze and optimize the performance of plausible policies. Our work falls in the latter category but the policies we employ are strongly motivated by the former.

Kimemia and Gershwin (1983) propose formal optimal control theory as an approach for studying general manufacturing flow control problems. They also propose a plausible class of control policies. The first formal proofs of optimality of a certain policy, however, were established later by Akella and Kumar (1986) and Bielecki and Kumar (1988) for a two machine state system with exponential up and down times, constant demand, linear holding and backorder costs. In particular, these papers establish the optimality of a hedging point policy: whenever up, the machine should produce at full capacity below a target inventory level called the hedging point and produce at a rate which enables it to stay there once it reaches this target.

There are several extensions of the hedging point type optimality structure. For instance, for a multiple machine state system with a Markovian transition structure, a multiple hedging point type policy is optimal: for each machine state where capacity exceeds demand rate, there is a corresponding hedging point; see Sethi et al. (2002) for related results. It is also known that with non-exponential machine state transitions, the structure of the optimal policy becomes more complex (see Hu and Xiang (1995) for an example) but hedging point policies remain useful due to their simplicity.

In addition, there is also a considerable literature on the performance analysis side. Sharifnia (1988) investigates the multiple machine state problem with Markovian transitions when a multiple hedging point policy is used and presents analytical results for the performance measures of the system. Finding the optimal hedging points, however, remains difficult. Liberopoulos and Hu (1995) also investigate the multiple machine state problem with Markovian transitions with a different focus. They establish useful monotonicity properties of the optimal hedging levels with respect to the machine state transition structure. We use these intuitive monotonicity properties for testing our procedure as part of our numerical experiments in Section 5.5.

There are also several other examples where hedging point type policies arise in single-part and single-stage systems under more general modeling assumptions (see Cassandras (2007) for a general model and its possible applications). A typical example is that of threshold subcontracting: additional manufacturing resources can be used depending on the inventory level. Tan (2002a), Hu et al. (2004), and Tan and Gershwin (2004) propose models and performance analysis approaches for this class of problems. Zhang et al. (2001) is another example in which joint production and marketing decisions are modelled through a hedging point type policy. Our approach, as we explain in more detail below, can be utilized in such settings with some straightforward modifications.

If one is concerned with a system with several machine and demand states and fairly general random disturbances (i.e., not necessarily exponentially distributed), then the generality of the problem and the intractability of an analytical solution make a simulation-based method an attractive approach. Using a simulation-based method for finding optimal hedging points has been tried before by Caramanis and Liberopoulos (1992), Liberopoulos and Caramanis (1994), Haurie et al. (1994), Yan et al. (1999), Brémaud et al. (1997), Yan et al. (1999), and Yin et al. (2001). However, the optimization side of all these papers were confined to the method of stochastic approximation and its variants, see Robbins and Monro (1951). Although very prominent, in practice these methods suffer from serious drawbacks such as slow convergence, lack of a good stopping criterion, and difficulty in enforcing feasibility. In particular, when there are constraints, these methods handle inequality constraints –even deterministic linear inequalities– via projection onto the feasible set. Such a projection operation can retard the performance of an optimization algorithm immensely, as illustrated by the simple example in Appendix 6 of Plambeck et al. (1996). In that example, such a method requires nearly 10^{43} steps to find the minimizer (the origin) of a linear function on the nonnegative orthant $\mathbf{R}^2_+$. Furthermore, even in the unconstrained case, the empirical performance of stochastic approximation type methods is very sensitive to the choice of a predetermined step size. Fu and Healy (1992), L'Ecuyer (1991), Glasserman and Tayur (1995), and Gürkan (2000) contain a number of examples which demonstrate this sensitivity.

These difficulties are partly reflected in the available numerical results of simulation-based optimization literature for finding optimal hedging points: none of the papers we are aware of (for single-part single-stage models) present numerical experiments with systems having more than two decision variables and they are all solving unconstrained problems. Furthermore, very often, comprehension of how the gradient estimates are derived is quite challenging for a non-specialist in gradient estimation literature since the arguments leading to a particular estimator are often done case by case. For example, Brémaud et al. (1997) is one of the papers developing a rigorous infinitesimal perturbation analysis (IPA) algorithm for a system whose transition times are not necessarily Markovian and the IPA analysis of the two hedging point case is done in a separate section than the IPA estimate for a single hedging point. Zhao and Melamed (2006, 2007) also deal with a general Make-to-Stock system (not necessarily Markovian) with random demand and production rates and IPA derivatives are computed for the time averaged inventory level and the time-averaged backorder level with respect to the base-stock level recursively via a sample-path analysis. However, derivation of their IPA derivatives is not straightforward for some initial conditions, which is also mentioned in Melamed et al. (2010) as a drawback and there are no numerical experiments reported in these papers.

Application of IPA derivatives in stochastic fluid models is an active area of research. It is of interest to develop computationally robust IPA formulas which are flexible (i.e., free of number of control parameters and initial conditions of the system) and directly computable from a sample path (see Wardi et al. (2010), Melamed et al. (2010) for recent work), which we attain with our approach as described below. Moreover, we also provide numerical experiments for various number of system control parameters and with service-level type probabilistic constraints, indicating the flexibility and practicality of our methodology.

To overcome the difficulties encountered in the literature, in this chapter we propose using a different simulation-based method, namely sample-path optimization, coupled with a generalized semi-Markov process (GSMP) representation for the system. As we will explain below, this has two consequences. On the optimization side, we are able to solve much larger (with several machine and demand states) and general (not necessarily exponential disturbances) problems even with probabilistic constraints. On the modeling and gradient estimation side, we derive a rigorous, intuitive, and transparent algorithm for a general system that can compute the function and gradient values of the objective function and service level type probabilistic constraints at any parameter setting in a single simulation run.

Next, we describe the basic ideas behind sample-path optimization. Roughly speaking, we are concerned with solving a problem of optimization involving a limit function f_∞ that we cannot observe. However, we can use simulation to observe functions f_n that converge pointwise to f_∞ as $n \to \infty$ almost surely. In the kind of applications typically encountered, f_∞ could be a steady-state performance measure of a dynamic system or an expected value in a static system. In systems that evolve over time, we can simulate the operation of the system for, say, n time units and then compute an appropriate performance measure. In static systems, we can repeatedly sample instances of the system and compute an average.

To be more precise, many problems in simulation-based optimization can be modelled by a real (or vector)-valued stochastic process $\{f_n(\omega,x) \mid n = 1,2,\ldots\}$. For each $n \geq 1$ and each $x \in \mathbf{R}^k$, $f_n(\omega,x)$ are random variables defined on a common probability space $(\Omega,\mathscr{F},P)$. It is helpful to keep in mind that x represents the decision variable, ω represents the sample path (which is indeed a list of random numbers that give rise to a particular discrete-event simulation run), and n represents the simulation length.

We assume the existence of a limit function $f_\infty(\omega,x)$ such that for every x, $f_n(\omega,x) \to f_\infty(\omega,x)$ as

$n \to \infty$ at almost every $\omega \in \Omega$. We are interested in finding a minimizer of f_∞ and in general we can only observe f_n for finite n. Therefore we will approximate minimizers of f_∞ using such information about f_n. The method is simple: fix an $\omega \in \Omega$ (by the method of common random numbers) and a large n (to get a good estimate of the limit function), compute a minimizer $x_n^*(\omega)$ of $f_n(\omega,\cdot)$, and take $x_n^*(\omega)$ as an approximate minimizer of $f_\infty(\omega,\cdot)$. Note that minimizers of $f_\infty(\omega,x)$ may generally depend on the sample path ω. However, in many practical problems for which one would anticipate using this technique, f_∞ is a deterministic function, for example a steady-state performance function or an expectation, i.e., it is independent of ω.

The basic case of sample-path optimization, which concerns the solution of simulation optimization problems with deterministic constraints, appeared in Plambeck et al. (1993) and Plambeck et al. (1996) and was analyzed in Robinson (1996). Robinson (1996) gave a number of sufficient conditions (on $\{f_n(\omega,x)\}$) that guarantee that if we take n large enough, each approximate minimizer $x_n^*(\omega)$ will be close to a true minimizer of the limit function. Plambeck et al. (1993) and Plambeck et al. (1996) reported extensive numerical experiments on large scale systems (project management involving PERT networks with up to 110 stochastic arcs and cycle time optimization in unreliable tandem production lines with up to 50 machines). In turn, Gürkan (2000) focused on optimization of buffer allocations in unreliable tandem production lines and reported results also for systems with up to 50 machines. Both Plambeck et al. (1996) and Gürkan (2000) used IPA for gradient estimation.

In the static case, a closely related technique centered around likelihood-ratio methods appeared in Rubinstein and Shapiro (1993) under the name of stochastic counterpart methods. The basic approach (and its variants) is also known as sample average approximation method in the stochastic programming literature; see for example Shapiro and Homem-De-Mello (1998) and Linderoth et al. (2006) in the static context of stochastic programming.

In Gürkan et al. (1996) and Gürkan et al. (1999a), the basic idea of using sample-path information was extended to solving stochastic variational inequalities and equilibrium problems. This work was used further in Gürkan et al. (1999b) for establishing almost-sure convergence of sample-path methods when dealing with general stochastic optimization problems with stochastic constraints. Furthermore, Birbil et al. (2006) deals with the theoretical analysis and application of sample-path methods to the so-called stochastic mathematical programs with equilibrium constraints.

In this chapter, we do not deal with theoretical issues related to convergence analysis; instead we solely focus on the operational issues that need to be addressed in solving the hedging point problem in practice. In general, doing a rigorous convergence analysis would require one to have additional insight on path-wise functional properties of the performance measures involved. This can be achieved if one is willing to make additional assumptions, say focus on certain distribution functions, as illustrated in Gürkan (2000). There, certain path-wise functional properties of sample throughput (as function of buffer capacities) are used to verify the sufficient conditions established in Robinson (1996) and almost sure convergence of sample-path optimization is proven. Here, we would like to work in full generality and we only verify the convergence of our procedure through numerical experiments.

There are two key points that make sample-path type methods attractive in practice: (i) once we fix a sample point ω and n, $f_n(\omega,x)$ becomes a deterministic function of x; (ii) IPA — when it is applicable — is able to compute *exact* gradients of $f_n(\omega,x)$. With these observations, very powerful methods of constrained and unconstrained deterministic optimization become available for use on the $f_n(\omega,\cdot)$. In the smooth case we can apply superlinearly convergent methods like the BFGS algorithm (or a variant for constrained problems) to minimize f_n to high accuracy in relatively few function and gradient evaluations. For more information on these algorithms see Fletcher (1987) and Gill et al. (1981), and for the software available see Moré and Wright (1993). Using superlinearly convergent methods enables us to be confident about the location and the accuracy of the minimizer of f_n, because we can differentiate between the errors due to the approximation of f_∞ by f_n and those due to the inaccurate computation of a minimizer of f_n. With slower algorithms like stochastic approximation, this is difficult if not impossible.

It is well-known that in many cases in which one uses a discrete event simulation, the exact partial derivatives (or directional derivatives) of $f_n(\omega,x)$ with respect to x can be obtained by using IPA in a single simulation run; see Glassermann (1991) and Ho and Cao (1991). To this end, we propose developing a GSMP representation of the system and utilizing that representation to derive recursive expressions for the exact derivatives. A GSMP can simply be thought as a mathematical framework which models the evolution of a discrete-event simulation. As it can be seen in Section 5.4, the GSMP framework enables us to develop an intuitive and transparent algorithm for a general system with multiple machine and demand states and with state transitions that are governed by general continuous distributions.

Finally, we would like to point out that our general approach, with appropriate modifications, could be applicable to several other challenging problems encountered in practice, especially considering the trends in using stochastic fluid models for decision making under uncertainty; see Liu and Gong (2002), Sun et al. (2004), and Cassandras (2007) for possible applications in finance, telecommunications, and engineering.

5.3 Problem Description

Consider a manufacturing system that produces a single product. The maximum production capacity of the system can take one of K possible values, r_i $(i=1,2,\ldots,K)$ according to a stationary process. We denote by $\alpha(t)(\alpha(t)=1,2,\ldots,K)$ the state of the machine at time t and $r_{\alpha(t)}$ the corresponding maximum production capacity in that state. The time spent in state i is a continuous random variable with a general distribution F_i. After state i, the next machine state k, is determined according to a matrix $\{P_{ik}\}$ with $\sum_{k=1}^{K} P_{ik}=1$.

The system has to cope with a random demand. Similar to machine states, we denote by $\beta(t)(\beta(t)=1,2,\ldots,M)$ the state of demand at time t and $d_{\beta(t)}$ the corresponding demand rate in that state. The time spent in demand state j also has a general distribution G_j and after state j, the next demand state k is determined according to a matrix $\{Q_{jk}\}$ with $\sum_{k=1}^{M} Q_{jk} = 1$. Note that although in many interesting problems occurring in practice $P_{ii} = Q_{jj} = 0$ would hold, we don't necessarily require it in our model.

Let $X(t)$ denote the inventory level at time t. Note that $X(t)$ can take negative values corresponding to backorders in the system. To this end, it is useful to distinguish the positive and negative parts of the inventory process. We define by $X^+(t) = X(t)I_{\{X(t)\geq 0\}}$ the production surplus of the system and by $X^-(t) = -X(t)I_{\{X(t)<0\}}$ the backorder level (where I_A is the indicator function corresponding to set A). The instantaneous cost function is then given as: $c^+X^+(t) + c^-X^-(t)$ where c^+ and c^- are, respectively the holding and backorder costs per item per unit time.

We can classify the combined states of the system as (i, j) $(i = 1, 2, \ldots, K, j = 1, 2, \ldots, M)$ considering the maximum production rates and demand rates simultaneously. A state (i, j) is called deficient if $r_i < d_j$. In such a state $X(t)$ has to be decreasing regardless of the production control selected. Similarly, state (i, j) is called non-deficient if $r_i \geq d_j$; in this case $X(t)$ can be increasing, decreasing, or constant depending on the actual production control. Actually, if $r_i = d_j$, $X(t)$ can be either decreasing or constant and such a state is called a zero state.

Let $v(t)$ $(r_{\alpha(t)} \geq v(t) \geq 0)$ denote the actual production rate at time t. The objective of the manufacturing flow control problem is to select $v(t)$ such that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T (c^+X^+(t) + c^-X^-(t))dt \tag{5.1}$$

is minimized. We study the above problem with the restriction that $v(t)$ is in the class of hedging point policies. A hedging point policy drives the system to the hedging point of the current system state at the fastest rate possible and then keeps it at the hedging point by adjusting the production rate until a system state change occurs. Accordingly, for each non-deficient system state (i, j), there is a hedging point z_{ij}. The control $v(t)$ is completely determined by the specification of z_{ij}s for non-deficient machine states:

$$v(t) = \begin{cases} r_i & \text{if } X(t) < z_{ij} \\ d_j & \text{if } X(t) = z_{ij} \\ 0 & \text{if } X(t) > z_{ij} \end{cases} \tag{5.2}$$

and in convention with the existing literature (see Liberopoulos and Hu (1995) for example) we assume that the production rate in deficient states is set to the maximum rate (i.e., $v(t) = r_i$) regardless of the inventory position. Naturally, other production rules can also be easily formulated within the same framework. The control problem under the restriction to this particular class of policies is to select the optimal values of the hedging points z_{ij} in order to minimize (5.1).

An important issue is the stability of such a system under which there exists a stationary solution. Let π_i $(i = 1, 2, \ldots, K)$ denote the stationary probability of being in machine state i and q_j $(j = 1, 2, \ldots, M)$ denote the stationary probability distribution of being in demand state j. Then a sufficient condition for stability of the system is given by:

$$\sum_{i=1}^{K} \pi_i r_i > \sum_{j=1}^{M} q_j d_j. \tag{5.3}$$

Brémaud et al. (1997) show that if condition (5.3) is not satisfied, then there exists no stationary solution and any transient solution goes to $-\infty$ as time goes to ∞ ($\lim_{t\to\infty} X(t) = -\infty$). We provide a proof of this in Appendix 5.A. Hereon, we will restrict our attention to systems satisfying condition (5.3).

In addition, we assume that the process $X(t)$ does not "hit and run" any of the hedging points. That is, there exist an $\varepsilon > 0$ such that if $t^{'}$ is a time that $X(t^{'})$ becomes z_{ij}, then we should have $X(t^{'} + \varepsilon) = z_{ij}$ for each i and j. Following the terminology of Brémaud et al. (1997), we will call this assumption "hit and stick". "Hit and stick" is almost always satisfied. In practice, it means that two events (end of holding time in a state and reaching to a hedging level) do not happen at the same instant, which is satisfied almost surely, since the probability of a continuous variable being equal to a specific value is always 0. Roughly speaking, excluding the possibility of the simultaneous occurrence of exogenous (end of holding time in a state) and endogenous (reaching to a hedging level) events helps to ensure the differentiability of the performance measure. Otherwise, one can still develop expressions for one-sided directional derivatives, but they may not be equal, as also noted by Brémaud et al. (1997) for a two hedging point system.

5.4 The GSMP and Gradient Estimation

As mentioned earlier, to carry out the subsequent simulation and infinitesimal perturbation analysis, we will utilize a generalized semi-Markov process (GMSP) representation of the system. Glynn (1989) provides a brief introduction and an excellent description of this framework can be found in Shedler (1993).

The basic idea of a GSMP can be explained as follows: There is a set of states and a set of events. The GSMP jumps from one state to another upon the occurrence of an event; at each state there are some active events. At any time, each active event is associated with a clock representing the residual lifetime of that event and a speed at which the clock runs down. If the clock corresponding to event e in state s equals k and the speed at which this clock runs is r, then e is scheduled to occur after k/r units of time. The next event and the time until it occurs are always determined by the smallest clock reading/clock speed ratio. More formally, let $k(e,t)$ be the reading of the clock for event e at time t and $r(e,t)$ be the speed at which that clock runs down. $E(t)$ is the current set of active events, i.e., the set of events with $r(e,t) \neq 0$. Then the next event to occur is given by

$$e^*(t) = \text{argmin}\left\{\frac{k(e,t)}{r(e,t)} | e \in E(t)\right\}.$$

Upon the occurrence of an event, changes may occur in the physical state, clock settings, and clock speeds: If event e occurs in state s, the process may move to a new state s' with a certain probability $p(s';s,e)$; the set of active events changes with the state; clocks for any old events which remain active continue to run in the new state; new clocks are initialized for all new active events and for the event which just occurred if it is also active in the new state. The initial value of each new clock for event e in state s is a random variable with a pre-specified cumulative distribution function $F(\cdot;s,e)$. This goes on until a termination criteria is reached.

To develop a GSMP representation for our system, let the current system states be $\alpha(t) = i$ and $\beta(t) = j$ and then define the following five possible events:

$\mathscr{M}\mathscr{F}$: the end of a sojourn time at current machine state i,

$\mathscr{U}\mathscr{H}$: reaching the hedging point z_{ij} of current system state (i, j) from below,

$\mathscr{D}\mathscr{H}$: reaching the hedging point z_{ij} of current system state (i, j) from above,

$\mathscr{D}\mathscr{G}$: the end of a sojourn time at current demand state j,

$\mathscr{T}_f$: the simulation termination.

We let $u(t) = v(t) - d_{\beta(t)}$ to denote the fill-in rate, the total rate of change in the inventory level process, at time t. Also let $W^M(t)$ denote the remaining time until the current machine state changes and $W^D(t)$ denote the remaining time until the current demand state changes. If we let T to denote the prespecified amount of time that the system will be simulated, we can then define the clock readings and machine speeds as in Table 5.1.

e	$\mathscr{M}\mathscr{F}$	$\mathscr{U}\mathscr{H}$	$\mathscr{D}\mathscr{H}$	$\mathscr{D}\mathscr{G}$	$\mathscr{T}_f$
$k(e,t)$	$W^M(t)$	$z_{ij} - X(t)$	$X(t) - z_{ij}$	$W^D(t)$	$T - t$
$r(e,t)$	1	$u(t)$	$-u(t)$	1	1

Table 5.1: The clock readings and associated speeds

Next, utilizing this GSMP representation, we derive a recursive formula to compute *exact* derivatives of long-run average cost per unit time with respect to hedging points, in a single simulation run. As mentioned earlier, we use IPA to compute derivatives of long-run average cost per unit time. Let $f : \mathbf{R}^{KM} \to \mathbf{R}$ be a function which has a gradient at a point $z = (z_1, \ldots, z_{KM})$, then IPA computes an array whose (i, j)th component is

$$\frac{\partial f(z)}{\partial z_{ij}} = \lim_{\Delta z \to 0} \frac{f(z + \Delta z y_{ij}) - f(z)}{\Delta z},$$

where $y_{ij} (i = 1, \ldots, K, j = 1, \ldots, M)$ is the $(i, j)th$ unit vector in $\mathbf{R}^{KM}$. In the following we use $\partial(\cdot)/\partial z_{ij}$ to denote the (i, j)th partial derivative of $(\cdot)$ at the point z. Furthermore, we abuse notation and use $z + \Delta z_{ij}$ and $z + \Delta z y_{ij}$ interchangeably; that is $\Delta z_{ij} = \Delta z y_{ij}$.

When $z = (z_{ij})$ is the vector of hedging points, as T gets large, the long-run average cost per unit time could be defined as $J_T(z) = C(T)/T$, where $C(T)$ is the cumulative cost incurred by the system over time

T as given by

$$C(T) = \int_0^T (c^+X^+(t) + c^-X^-(t))dt. \tag{5.4}$$

Note that $C(T)$ depends on the production rate $v(t)$ and the inventory level $X(t)$ which which are adjusted in each state depending on the hedging level z. Let $t_0, t_1, \ldots$ be the event occurrence times in a sample path and $\tau_n = t_n - t_{n-1}$ be the time between the $(n-1)$st and nth events. Without loss of generality we assume $t_0 = 0$. If the nth event is the event that the system has been working T time units, i.e., $t_n = T$ for some n, we have

$$\frac{\partial J_T(z)}{\partial z_{ij}} = \frac{\partial}{\partial z_{ij}}(\frac{C(T)}{T}) = \frac{\partial C(T)}{\partial z_{ij}}\frac{1}{T} = \frac{\partial C(t_n)}{\partial z_{ij}}\frac{1}{t_n}. \tag{5.5}$$

From (5.5) we see that $J_T(z)$ has derivatives if and only if $C(t_n)$ has them. Next we show that $C(t_n)$ has the desired property and derive a recursive expression for $\partial C(t_n)/\partial z_{ij}$, which is a quantity computable from the simulation information generated up time t_n.

For $n = 0, 1, \ldots$ let e_{n+1} be the $(n+1)$st event. Then

$$e_{n+1} = e^*(t_n) := \text{argmin}\left\{\frac{k(e,t_n)}{r(e,t_n)}|e \in E(t_n)\right\},$$

and

$$t_{n+1} = t_n + \tau_{n+1} \qquad \text{where} \qquad \tau_{n+1} = \frac{k(e_{n+1},t_n)}{r(e_{n+1},t_n)}. \tag{5.6}$$

Recall that $v(t_n)$ is the actual production rate and $u(t_n)$ is the fill-in rate at t_n; that is, if the controller is in state (i, j) at t_n, then $u(t_n) = v(t_n) - d_j$. (5.7) below gives a recursive expression for $C(t_{n+1})$.

$$\begin{aligned} C(t_{n+1}) \quad = \quad & C(t_n) + \\ & I_{\{X(t_n)\geq 0\}}I_{\{X(t_{n+1})\geq 0\}}c^+[X(t_{n+1})\tau_{n+1} - \frac{u(t_n)\tau_{n+1}}{2}\tau_{n+1}] + \\ & I_{\{X(t_n)<0\}}I_{\{X(t_{n+1})<0\}}c^-[-X(t_{n+1})\tau_{n+1} + \frac{u(t_n)\tau_{n+1}}{2}\tau_{n+1}] + \\ & I_{\{X(t_n)\geq 0\}}I_{\{X(t_{n+1})<0\}}\{c^+[\frac{-X(t_n)}{2}\frac{X(t_n)}{u(t_n)}] + c^-[\frac{-X(t_{n+1})}{2}\frac{X(t_{n+1})}{u(t_n)}]\} + \\ & I_{\{X(t_{n+1})\geq 0\}}I_{\{X(t_n)<0\}}\{c^+[\frac{X(t_{n+1})}{2}\frac{X(t_{n+1})}{u(t_n)}] + c^-[\frac{X(t_n)}{2}\frac{X(t_n)}{u(t_n)}]\}. \end{aligned} \tag{5.7}$$

Here is the main theorem.

Theorem 5.4.1. *For $n \geq 0$, $C(t_{n+1})$ has partial derivatives at z for all z and $i = 1, \ldots, K$, $j = 1, \ldots, M$,* *given by*

$$
\begin{aligned}
\frac{\partial C(t_{n+1})}{\partial z_{ij}} = & \frac{\partial C(t_n)}{\partial z_{ij}} + \\
& I_{\{X(t_n)\geq 0\}} I_{\{X(t_{n+1})\geq 0\}} c^+ [\frac{\partial X(t_{n+1})}{\partial z_{ij}} \tau_{n+1} + \frac{\partial \tau_{n+1}}{\partial z_{ij}} X(t_{n+1}) - u(t_n)\tau_{n+1} \frac{\partial \tau_{n+1}}{\partial z_{ij}}] + \\
& I_{\{X(t_n)< 0\}} I_{\{X(t_{n+1})< 0\}} c^- [-\frac{\partial X(t_{n+1})}{\partial z_{ij}} \tau_{n+1} - \frac{\partial \tau_{n+1}}{\partial z_{ij}} X(t_{n+1}) + u(t_n)\tau_{n+1} \frac{\partial \tau_{n+1}}{\partial z_{ij}}] + \\
& I_{\{X(t_n)\geq 0\}} I_{\{X(t_{n+1})< 0\}} \{c^+ [\frac{-X(t_n)}{u(t_n)} \frac{\partial X(t_n)}{\partial z_{ij}}] + c^- [\frac{-X(t_{n+1})}{u(t_n)} \frac{\partial X(t_{n+1})}{\partial z_{ij}}]\} + \\
& I_{\{X(t_{n+1})\geq 0\}} I_{\{X(t_n)< 0\}} \{c^+ [\frac{X(t_{n+1})}{u(t_n)} \frac{\partial X(t_{n+1})}{\partial z_{ij}}] + c^- [\frac{X(t_n)}{u(t_n)} \frac{\partial X(t_n)}{\partial z_{ij}}]\}.
\end{aligned}
\tag{5.8}
$$

The proof of Theorem 5.4.1 is by induction. In practice, to compute $\partial C(t_{n+1})/\partial z_{ij}$ we need to be able to compute $\partial X(t_{n+1})/\partial z_{ij}$ (done in Lemma 5.4.4 below), $\partial \tau_{n+1}/\partial z_{ij}$ (done in Lemmas 5.4.2, 5.4.6, 5.4.5, 5.4.7 below), and $\partial u(t_n)/\partial z_{ij}$ (done in Lemma 5.4.3 below) recursively, using information that is available up to t_{n+1}. Before proceeding with the proof of the theorem, we first mention the concept of "similarity" (which we address in detail in Appendix 5.B) and state a few technical lemmas. The proofs of all lemmas of this section are in Appendix 5.C.

"Similarity" of the nominal path and the perturbed path (which are made precise in Appendix 5.B) is a standard issue one needs to deal with when developing IPA algorithms. In Appendix 5.B, we show that along any sample path of finite length, say k events, with $\min\{\tau_n | n = 1,2,\ldots,k\} > 0$, there is always a $\Delta z_{ij} > 0$ (or $\Delta z_{ij} < 0$ whose size depends on the sample path) small enough such that increasing (or decreasing, respectively) z by Δz_{ij} does not cause any event order change; that is, the perturbed path and the nominal path remain similar.

Lemma 5.4.2. *For all $n = 0,1,\ldots$, $r(e_{n+1},t_n)$ has partial derivatives at z for all z and $i = 1,\ldots,K$, $j = 1,\ldots,M$, which are 0.*

Proof. See Appendix 5.C.

Lemma 5.4.3. *For all $n = 0,1,\ldots$, $u(t_n)$ has partial derivatives at z for all z and $i = 1,\ldots,K$, $j = 1,\ldots,M$, which are 0.*

Proof. See Appendix 5.C.

Lemma 5.4.4. *Suppose that $X(t_{n-1})$ and τ_n have partial derivatives at z for all z and $i = 1,\ldots,K$, $j =$*

$1,\ldots,M$. Then $X(t_n)$ has partial derivatives at z for all z and $i=1,\ldots,K$, $j=1,\ldots,M$ given by

$$\frac{\partial X(t_n)}{\partial z_{ij}} = \frac{\partial X(t_{n-1})}{\partial z_{ij}} + u(t_{n-1}) \cdot \frac{\partial \tau_n}{\partial z_{ij}}.$$

Proof. See Appendix 5.C.

Lemma 5.4.5. *Suppose that $W^M(t_{n-1})$, $W^D(t_{n-1})$, and τ_n have partial derivatives at z for all z and $i=1,\ldots,K$, $j=1,\ldots,M$. Then $W^M(t_n)$ and $W^D(t_n)$ have partial derivatives at z for all z and $j=1,\ldots,K$ given by*

$$\frac{\partial W^M(t_n)}{\partial z_{ij}} = \frac{\partial W^M(t_{n-1})}{\partial z_{ij}} - \frac{\partial \tau_n}{\partial z_{ij}}$$

and

$$\frac{\partial W^D(t_n)}{\partial z_{ij}} = \frac{\partial W^D(t_{n-1})}{\partial z_{ij}} - \frac{\partial \tau_n}{\partial z_{ij}}.$$

Proof. See Appendix 5.C.

Lemma 5.4.6. *Suppose that $X(t_n)$, $W^M(t_n)$, $W^D(t_n)$, and t_n have partial derivatives at z for all z and $i=1,\ldots,K$, $j=1,\ldots,M$. Then $k(e_{n+1},t_n)$ has partial derivatives at z for all z and $i=1,\ldots,K$, $j=1,\ldots,M$ given by*

$$\frac{\partial k(e_{n+1},t_n)}{\partial z_{ij}} = \begin{cases} \frac{\partial W^M(t_n)}{\partial z_{ij}} & \text{if } e_{n+1} = \mathscr{MF}, \\ I_{\{i=\alpha(t_n),j=\beta(t_n)\}} - \frac{\partial X(t_n)}{\partial z_{ij}} & \text{if } e_{n+1} = \mathscr{UH}, \\ \frac{\partial X(t_n)}{\partial z_{ij}} - I_{\{i=\alpha(t_n),j=\beta(t_n)\}} & \text{if } e_{n+1} = \mathscr{DH}, \\ \frac{\partial W^D(t_n)}{\partial z_{ij}} & \text{if } e_{n+1} = \mathscr{DG}, \\ -\frac{\partial t_n}{\partial z_{ij}} & \text{if } e_{n+1} = \mathscr{T}_f. \end{cases} \tag{5.9}$$

Proof. See Appendix 5.C.

Lemma 5.4.7. *Suppose that t_n has partial derivatives at z for all z and $i=1,\ldots,K$, $j=1,\ldots,M$. Then t_{n+1} has partial derivatives at z for all z and $i=1,\ldots,K$, $j=1,\ldots,M$ given by*

$$\frac{\partial t_{n+1}}{\partial z_{ij}} = \frac{\partial t_n}{\partial z_{ij}} + \frac{\partial \tau_{n+1}}{\partial z_{ij}} \tag{5.10}$$

where

$$\frac{\partial \tau_{n+1}}{\partial z_{ij}} = \frac{1}{r(e_{n+1},t_n)} \frac{\partial k(e_{n+1},t_n)}{\partial z_{ij}}. \tag{5.11}$$

Proof. See Appendix 5.C.

The next lemma will start the induction.

Lemma 5.4.8. *$t_1, \tau_1, X(t_0), W^M(t_0), W^D(t_0)$, and $C(t_0)$ have partial derivatives at z for all z and $i = 1,\dots,K$, $j = 1,\dots,M$. These are given by, for all i and j:*

$$\frac{\partial X(t_0)}{\partial z_{ij}} = \frac{\partial W^M(t_0)}{\partial z_{ij}} = \frac{\partial W^D(t_0)}{\partial z_{ij}} = \frac{\partial C(t_0)}{\partial z_{ij}} = 0 \textit{ and}$$

$$\frac{\partial t_1}{\partial z_{ij}} = \frac{\partial \tau_1}{\partial z_{ij}} = \frac{1}{r(e_1,t_0)} \frac{\partial k(e_1,t_0)}{\partial z_{ij}} = \begin{cases} \frac{1}{u(t)} I_{\{i=\alpha(t_0),\, j=\beta(t_0)\}} & \textit{if } e_1 = \mathscr{U}\mathscr{H}, \\ \frac{1}{u(t)} I_{\{i=\alpha(t_0),\, j=\beta(t_0)\}} & \textit{if } e_1 = \mathscr{D}\mathscr{H}, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. See Appendix 5.C.

Now we are ready to prove the main theorem.

Proof of Theorem 5.4.1. First, observe that because of Remark 5.B.4 (in Appendix 5.B), we can treat $I_{\{X(t_n(z))\geq 0\}}$, $I_{\{X(t_{n+1}(z))\geq 0\}}$, $I_{\{X(t_n(z))<0\}}$, and $I_{\{X(t_{n+1}(z))<0\}}$ as constants while using the chain rule of differentiation. Next, Lemma 5.4.8 provides the start of inductive argument. Suppose that $t_n, \tau_n, X(t_{n-1}), W^M(t_{n-1}), W^D(t_{n-1})$, and $C(t_{n-1})$ have partial derivatives at z for all z and $i = 1,\dots,K$, $j = 1,\dots,M$. Apply Lemma 5.4.5 to get the recursions for $W^M(t_n)$ and $W^D(t_n)$; Lemma 5.4.4 for $X(t_n)$; and Lemma 5.4.7 for t_{n+1} and τ_{n+1}. Lemmas 5.4.2 and 5.4.6 provide the necessary information about $\partial k(e_{n+1},t_n)/\partial z_{ij}$ and $\partial r(e_{n+1},t_n)/\partial z_{ij}$. ∎

In Appendix 5.D, we provide a pseudo code for our simulation and an IPA algorithm that we developed utilizing our GSMP representation and the recursions developed in this chapter. The steps (needed for the IPA algorithm) added to the basic simulation algorithm are marked as "IPA".

Note that the GSMP framework together with the recursive manner of computing partial derivatives allows one to use IPA also for the computation of second-order partial derivatives. However, we have not pursued this here, since most of the state-of-art deterministic nonlinear optimization codes (including E04UCC we use in our numerical experiments reported in Section 5.5) are capable of using just the derivative information for maintaining and utilizing efficiently dynamic BFGS quasi-Newton approximations for the Hessian.

It is also worth noting that an important issue in gradient estimation literature has been the development of estimators with good asymptotic behavior. The convergence theorems and convergence rate results for stochastic approximation type methods are concerned with the unbiasedness of the gradient estimator, for example see Kushner and Clark (1978); whereas convergence theorems for sample-path optimization may not have these requirements in general, for example see Robinson (1996). When using sample-path optimization the main requirement from the practical point of view is an *exact* gradient or a directional derivative (whichever is available) of the sample function to be optimized so that an efficient deterministic optimizer is able to utilize this sensitivity information. In this sense, the sample-path performance functions (for finite time) of the buffer optimization problem addressed in Gürkan (2000) constitute an interesting example. They were discontinuous functions and their exact derivatives were most likely biased. However, under the assumption that the steady-state throughput is a continuous function of buffer capacities, Gürkan (2000) proves the almost sure convergence of sample-path optimization. Furthermore, the deterministic optimizer was able to utilize the *exact* directional derivatives of the discontinuous sample functions efficiently and find the optimal buffer allocations in systems with up to 50 unreliable machines.

Proving the unbiasedness of the estimator we derived here is rather standard and we tackle that next.

Lemma 5.4.9. *For any z and $i = 1,\dots,K,\ j = 1,\dots,M$, $|\frac{J_T(\omega,z+\Delta z_{ij})-J_T(\omega,z)}{\Delta z_{ij}}| \leq (c^+ + c^-)$, for any Δz_{ij} small enough so that the event order list does not change, where ω is the sequence of random numbers which give rise to a particular simulation run.*

Proof. See Appendix 5.C.

Theorem 5.4.10. *IPA provides an unbiased gradient estimator; that is, we have $E[\frac{\partial J_T(\omega,z)}{\partial z_{ij}}] = \frac{\partial E[J_T(\omega,z)]}{\partial z_{ij}}$.*

Proof: From Lemma 5.4.9, we have $|\frac{J_T(\omega,z+\Delta z_{ij})-J_T(\omega,z)}{\Delta z_{ij}}| \leq (c^+ + c^-)$. From Theorem 5.4.1,

$$\frac{\partial J_T(\omega,z)}{\partial z_{ij}} = \lim_{\Delta z_{ij}\to 0} \frac{J_T(\omega,z+\Delta z_{ij}) - J_T(\omega,z)}{\Delta z_{ij}}$$

exists at any z with probability 1.

We choose $g(\omega) := (c^+ + c^-)$ as the dominating function and use Lebesgue's dominated convergence theorem to obtain

$$
\begin{aligned}
\frac{\partial E[J_T(\omega,z)]}{\partial z_{ij}} &= \lim_{\Delta z_{ij}\to 0}\int_{\Omega}\frac{J_T(\omega,z+\Delta z_{ij})-J_T(\omega,z)}{\Delta z_{ij}}dP \\
&= \int_{\Omega}\lim_{\Delta z_{ij}\to 0}\frac{J_T(\omega,z+\Delta z_{ij})-J_T(\omega,z)}{\Delta z_{ij}}dP \\
&= E[\frac{\partial J_T(\omega,z)}{\partial z_{ij}}]. \blacksquare
\end{aligned}
$$

5.5 Numerical Experiments

In this section, we report some of our numerical experiments used for testing the empirical performance of the ongoing methodology. First, we compared the analytical results of some production control problems found in the literature with our results. Then, we solved five different examples with dimensions up to ten machine and four demand states. In Example 5.5.1, we also compared sample-path optimization (SPO) with a variant of stochastic approximation (SA), called single-run optimization (SRO) (see Meketon (1987)), since the SRO variant of SA is reported to perform better than classical SA for some problems in the literature; see for example Leung (1990). In all numerical experiments, we used the exponential distribution for state transition times. Essentially any continuous distribution whose support is on $(0,\infty)$ can be used; the choice of the exponential distribution is mainly made for comparison and convenience.

In order to simulate a general model with K machine and M demand states, we use our pseudo code written in C which calculates the gradients and the value of the objective function and constraints in a single simulation run. We convey these values to the optimizer which determines an "optimal" point. The illustration of the method is given in Figure 2.2 (b) of Chapter 2 (x corresponds to the hedging points z in this chapter). We formulate the approximating problem as an NLP problem in the optimizer and the average cost function in the objective is defined as an external nonlinear function. We use an iterative procedure in which NLP solver calls the simulation model (GSMP) at any decision point that it would like to evaluate its optimality. The GSMP model enables us to obtain the value and the (directional) derivatives of the approximating average cost function at a single simulation run at each iteration. The iterations continue until the optimality conditions of the NLP problem at a decision point z^* are satisfied, which is taken as the optimal threshold levels of the system. In Section 5.5.3, we also include stochastic constraints in the NLP problem and use sample-path method in a similar way to solve the approximating problem. In the modified stochastic problem, we constrain the long run proportion of time that the system is in backlog state below a pre-specified level.

As the optimizer, we use the deterministic nonlinear optimization code E04UCC of NAG C library, Mark 7, NAG (2002). E04UCC is designed to minimize an arbitrary smooth function subject to constraints, which may include simple bounds on the variables, linear constraints, and smooth nonlinear constraints. Essentially, it is a sequential quadratic programming method incorporating an augmented Lagrangian merit function and a BFGS quasi-Newton approximation to the Hessian of the Lagrangian. The code iteratively determines the total number of simulation runs N required to find an approximate minimizer; this is controlled by the "Optimality Tolerance $(OptTol)$". The parameter $OptTol$ specifies the accuracy to which the user wishes the final iterate to approximate a solution of the problem. Broadly speaking, $OptTol$ indicates the number of correct figures desired in the objective function at the solution. For example, if $OptTol$ is 10^{-6}, the final value of the objective function would have approximately six correct figures. When there are only linear constraints, E04UCC considers a point "optimal" if the current step length, the norm of the search direction, and the norm of the projected gradient become sufficiently small. In the presence of nonlinear constraints, to be considered "optimal", the point should also satisfy an additional stopping criteria; namely, the violation of all active constraints should also become sufficiently small, see NAG (2002) for additional details. It is worth noting that in the absence of convexity of the functions involved, a nonlinear optimization program like E04UCC (rather than a specialized global optimization code) can only guarantee to find a local optimum which may or may not be a global optimum. The convexity of the cost function is proven by Akella and Kumar (1986) for a two machine state (up and down) system in the Markovian case. In systems with multiple hedging points, counterexamples show that convexity does not hold in general as we also observe in our numerical experiments.

As mentioned earlier, to verify the correctness and accuracy of our numerical procedures, we first compared our results with other results we could find in the literature. These include analytical results reported in Bielecki and Kumar (1988) and Gershwin (1994), and analytical/numerical results reported in Haurie et al. (1994) (their single-part model), Yan et al. (1994) (their single-stage model), and Liberopoulos and Caramanis (1994) (their single-part model). Aside from the system of Liberopoulos and Caramanis (1994) which had two hedging points, all the others were systems with a single hedging point. In all cases, we observed that our results converged to the correct optimal solutions with relatively little computational effort.

Additional test cases are provided by Liberopoulos and Hu (1995). They analytically established the relative ordering of optimal hedging points of more than two machine states in four specific Markov chain systems. In our experiments, we used their Systems 1, 2, and 3 with different numbers of machine states and parameter values. Optimal hedging points that we found had always the same ordering properties established by Liberopoulos and Hu (1995).

All the runs reported here are performed by a Dell PC with Dual-Intel Xeon, 2.66 GHz processors, and 2GB 266 MHz DDR Non-ECC SDRAM Memory using Windows 2000.

5.5.1 Optimization of Unconstrained Production Control Problems

In this part, we report the results of three numerical experiments. An additional test for the quality of the solutions we compute in these experiments is provided through a well-known result: the probability that the system is in backlog state ($P(X(t) < 0)$) at an optimal solution should be equal to the proportion $c^+/(c^+ + c^-)$ (see Brémaud et al. (1997)). In our computations, we approximate the probability that the system is in backlog state by computing the long run proportion of time that the system is in backlog state which we denote by $P_{blog}(z)$.

Example 5.5.1. The first experiment is with four machine states and a constant demand rate. Table 5.2 contains the data of this problem and the corresponding transition matrix of the four machine states is in Figure 5.1. Note that $P_{blog}(z^*_\infty)$ should be equal to 0.090910 at the optimal hedging point z^*_∞.

In this example, we have two non-deficient machine states, namely 3 and 4. We used $T = 500,000$ simulation time per simulation call which corresponded to around $5,000,000$ machine events in order to get a good estimate of the limit function. We set $OptTol$ to 10^{-6} and solved the problem from five different initial points A, B, C, D, and E; we observed convergence to the same final point z^*_T with $P_{blog}(z^*_T) = 0.090910$. As we do not know the analytical solution, we found the "optimal" solution z^*_∞ and the function values $J_\infty(z)$ by using SPO with a very large computational effort; using a simulation time of $5,000,000$ per simulation call which corresponded to around 50,000,000 machine events. We observed that z^*_∞ and z^*_T are very close with $P_{blog}(z^*_\infty) = P_{blog}(z^*_T) = 0.090910$, which confirms their optimality. The results are summarized in Table 5.3. Recall that N is the number of simulation runs determined by E04UCC to find an approximate minimizer.

Machine states	1	2	3	4
Maximum production rate	0	4	80	40

d	c^+	c^-	$c^+/(c^+ + c^-)$
9.533	1	10	0.090910

Table 5.2: Specifications of 4-machine state system with constant demand rate for Example 5.5.1

$$\begin{bmatrix} -4 & 4 & 0 & 0 \\ 0 & -8 & 0 & 8 \\ 14 & 0 & -15 & 1 \\ 0 & 1 & 9 & -10 \end{bmatrix}$$

Figure 5.1: Transition rate matrix of machine states for Example 5.5.1

As mentioned earlier, for comparison, we also solved this example by using an SRO algorithm of the form: $z_{n+1} = \Pi_\Phi(z_n - \frac{a_0}{n} g_n)$, where z_n is the hedging point vector at the nth iteration, g_n is the gradient estimate at z_n, a_0 is the predetermined step size constant, and $\Pi_\Phi(\cdot)$ is the projection onto the feasible set Φ determined by the simple bound constraints. It is well-known that it is possible for an SRO algorithm to diverge if one does not impose explicit bounds on the variables to ensure the boundedness of the iterates. We used a lower bound with each element equal to zero and an upper bound with each element equal to

1000.

We tried to observe the behavior of SRO iterates under the same computational budget used by SPO. In our implementation of the SRO, for each initial point, we used $T = 100,000$ simulation time per simulation call which corresponded to around $1,000,000$ machine events and we made $5N$ iterations (i.e., simulation runs), each run corresponding to one iteration of the algorithm. We chose to have a runlength of approximately $1,000,000$ machine events per iteration since Liberopoulos and Caramanis (1994) used this value in their numerical experiments with the stochastic approximation method. Since the simulation time T used in each iteration of the SRO is one fifth of the simulation time T used in each iteration of SPO, we made $5N$ iterations for SRO so that we could compare both methods under the same computational budget. Table 5.4 contains the results of the SRO algorithm with different a_0 values; the *"relative error"* is calculated as $\left|\frac{J_\infty(z)-J_\infty(z^*_\infty)}{J_\infty(z^*_\infty)}\right|$.

As mentioned earlier, it is well-known that one of the main drawbacks of SA-type algorithms (including the SRO variant) is the sensitivity of their empirical performance to the choice of a_0; documented, for example, in Fu and Healy (1992), L'Ecuyer et al. (1994), Glasserman and Tayur (1995), and Gürkan (2000). Our own experiments also confirmed this sensitivity issue even in this small example and we spent considerable effort to find reasonable values of a_0 which also depend on the initial point. For example, as seen in Table 5.4, from initial points A, B, C, and D, the iterates converge to the optimal solution with a_0 equal to 20; whereas from initial point E, among the ones we tried, the only good value of a_0 is 1.8.

Experimenting with several a_0s was feasible in this example since this was a small problem (with a relative short run time, no constraints, and two decision variables) and we knew where the "optimal" solution was. Since SRO is a first-order gradient method, one would expect to experience the same difficulties more severely in larger examples. Therefore we decided not to compare SRO with SPO in larger unconstrained examples of Section 5.5.1. For the constrained examples of Section 5.5.3, since we are not aware of any literature addressing how an SA-type method can handle those, we did not perform comparisons there either.

Machine states	3	4	$J_T(z)$	$P_{blog}(z)$	$J_\infty(z)$
Initial point *A*	5.00	5.00	8.398	0.176439	8.376
Initial point *B*	20.00	20.00	17.360	0.001518	17.359
Initial point *C*	5.00	20.00	11.177	0.063795	11.163
Initial point *D*	20.00	5.00	12.764	0.021478	12.742
Initial point *E*	1.00	2.00	18.794	0.576287	18.769
Final point(z_T^*)	7.44	5.27	7.382	0.090910	7.357
“Optimum” point(z_∞^*)	7.45	5.25	7.382	0.090910	7.357

Initial point	N	Total CPU time
A	14	1 min 27 sec
B	14	1 min 27 sec
C	11	1 min 06 sec
D	11	1 min 06 sec
E	13	1 min 21 sec

Table 5.3: Optimal solution and the run-time information for Example 5.5.1

Initial Point	Final point (z_3, z_4) $J_\infty(z)$ *"relative error"*								Number of iterations (5N)
	$a_0 = 1.5$	$a_0 = 1.8$	$a_0 = 2$	$a_0 = 10$	$a_0 = 15$	$a_0 = 20$	$a_0 = 30$	$a_0 = 40$	
A	(7.13, 5.43) 7.371 0.0019	(7.23, 5.50) 7.366 0.0012	(7.28, 5.53) 7.365 0.0011	(7.45, 5.59) 7.363 0.0008	(7.46, 5.35) 7.358 0.0001	(7.45, 5.27) 7.357 0.0000	(7.45, 5.23) 7.357 0.0000	(7.45, 5.23) 7.357 0.0000	70
B	(15.91, 17.08) 17.786 1.4176	(15.19, 16.44) 13.143 0.7865	(14.72, 16.01) 12.724 0.7295	(7.45, 6.55) 7.432 0.0102	(7.45, 5.28) 7.357 0.0000	(7.45, 5.23) 7.357 0.0000	(7.45, 5.23) 7.357 0.0000	(6.53, 10.37) 8.054 0.0947	70
C	(5.30, 17.64) 10.343 0.4059	(5.36, 17.17) 10.181 0.3838	(5.40, 16.86) 10.075 0.3694	(7.20, 7.78) 7.565 0.0283	(7.46, 5.64) 7.365 0.0011	(7.46, 5.26) 7.357 0.0000	(7.46, 5.24) 7.357 0.0000	(7.47, 5.24) 7.357 0.0000	55
D	(16.12, 4.12) 10.445 0.4197	(15.41, 4.02) 10.074 0.3693	(14.95, 3.97) 9.843 0.3379	(7.48, 5.15) 7.358 0.0001	(7.45, 5.23) 7.357 0.0000	(7.46, 5.24) 7.357 0.0000	(7.33, 7.62) 7.545 0.0255	(3.43, 42.22) 18.713 1.54363	55
E	(7.17, 4.92) 7.372 0.0020	(7.44, 5.21) 7.357 0.0000	(7.59, 5.36) 7.360 0.0004	(13.46, 7.42) 9.618 0.3073	(17.25, 10.21) 12.484 0.6969	(20.78, 14.12) 15.953 1.1684	(27.32, 23.25) 23.422 2.1836	(32.57, 33.84) 30.326 3.1221	65

Table 5.4: Solutions generated by SRO with different a_0 values

Example 5.5.2. We modify the system in Example 5.5.1 by adding four demand states with a transition rate matrix as in Figure 5.2. The corresponding demand rates are given in Table 5.5. Note that unit holding and backorder costs are the same, thus $P_{blog}(z^*_\infty)$ should still be equal to 0.090910 at the optimal hedging point z^*_∞.

In this example, we have eight non-deficient states and it can be seen that the system copes with a random demand which has an average value equal to the constant demand of Example 5.5.1; that is 9.533. We used $T = 1,000,000$ simulation time per simulation call which corresponded to around $12,000,000$ events in order to get a good estimate of the limit function. We again set $OptTol$ to 10^{-6}, solved the problem from different initial points, and observe convergence to the same point with $P_{blog}(z^*_T) = 0.090910$, which confirms its optimality. Table 5.6 displays the system cost and the probability that the system is in backlog state at both the initial points and the optimal point.

If we compare the hedging point levels in Examples 5.5.1 and 5.5.2, there are significant differences. For instance, in Example 5.5.1, the optimal hedging level for machine state 4 is 5.25 whereas, in Example 5.5.2, it differs between 3.96 and 22.15 depending on the demand state. A similar result is also observed for machine state 3. The differences between the optimal hedging levels in Examples 5.5.1 and 5.5.2 clearly illustrate that we cannot ignore the stochastic behavior of demand; that is, the variability of demand has an important effect on optimal hedging points. Therefore considering only the average demand without paying attention to the variability of demand leads to incorrect decisions.

$$\begin{bmatrix} -2 & 2 & 0 & 0 \\ 7.5 & -8 & 0.5 & 0 \\ 0 & 0.3 & -3 & 2.7 \\ 0 & 0 & 5 & -5 \end{bmatrix}$$

Figure 5.2: Transition rate matrix of demand states for Example 5.5.2

Demand states	1	2	3	4
Rate	5	8	15	20

Table 5.5: Demand rates for Example 5.5.2

Example 5.5.3. The third experiment is on a system with ten machine and four demand states. Table 5.7 contains the data of the problem and the transition rate matrix of machine and demand states are shown in Figure 5.3. This system has twenty two non-deficient states. We used $T = 3,000,000$ simulation time per simulation call which corresponded to around $25,000,000$ events.

Here, different from Examples 5.5.1 and 5.5.2, when we started from different initial points, we observed finding different "optimal" solutions. To make sure that these differences were not caused by insufficient accuracy, we increased $OptTol$ to 10^{-7}, but still observed the same behaviour as reported

Machine states	3	3	3	3	4	4	4	4		
Demand states	1	2	3	4	1	2	3	4	$J_T(z)$	$P_{blog}(z)$
Initial point A	5.00	5.00	5.00	5.00	5.00	5.00	5.00	5.00	28.766	0.261616
Initial point B	25.00	25.00	25.00	25.00	25.00	25.00	25.00	25.00	24.390	0.036370
z_T^*	6.72	9.67	24.16	24.78	3.96	7.90	21.23	22.15	17.030	0.090910

Initial point	N	Total CPU time
A	16	7 min 07 sec
B	32	13 min 44 sec

Table 5.6: Optimal solution and the run-time information for Example 5.5.2

in Table 5.8; different initial points are indicated as A, B, and C and different final solutions reached are indicated as z_A^*, z_B^*, and z_C^*, respectively. In general, there could be two major sources for the multiplicity of solutions. First, when an objective function is rather flat near an optimum point (as observed in this example and discussed in Section 5.5.2.5), there may be several points that can be declared as "optimum" for all practical purposes; a situation termed as "multiple optima". Clearly, all the relevant performance measures would take equal (or very close) values at all of these optimum points. Second, in the absence of (strict) convexity, a function can have several local optima, which may have similar or different performances. We also discuss the nonconvex nature of this objective function in more detail in Section 5.5.2.6. It is worth observing that all the different solution points in Table 5.8 have very similar performance measures; that is, the system cost does not differ significantly and the probability that the system is in backlog state is very close to 0.090910 for all the solution points suggesting multiple optima.

Furthermore, it can also be seen from Table 5.8 that some hedging points are close to each other and form clusters. We also discuss this clustering effect in more detail in Section 5.5.2.7.

Machine states	1	2	3	4	5	6	7	8	9	10
Maximum production rate	0.4	0	0.8	8	5	1.2	10	6	3	15

Demand states	1	2	3	4
Rate	1.3	1.8	3.2	4.0

c^+	c^-	$c^+/(c^+ + c^-)$
1	10	0.090910

Table 5.7: Specifications of 10-machine and 4-demand state system for Example 5.5.3

$$\begin{bmatrix} -12 & 0 & 0 & 4 & 4 & 4 & 0 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 & 0 & 0 & 2 & 2 & 1 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 7 & 0 & 0 & -8 & 0 & 0 & 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 & -6 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & 0 & -5 & 0 & 0 \\ 0 & 0.9 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0.1 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 2 & -5 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 2 & 0 & 0 \\ 7.9 & -8 & 0.1 & 0 \\ 0 & 0.1 & -5 & 4.9 \\ 0 & 0 & 10 & -10 \end{bmatrix}$$

Figure 5.3: Transition rate diagrams of machine and demand states for Example 5.5.3

5.5.2 Observations and Conjectures From Numerical Experiments from Unconstrained Systems

In these numerical experiments, we have undertaken many simulation runs and observed some behavior that we found interesting during these runs. Below, we report some of these as observations and conjectures.

5.5.2.1 Effect of Load

As mentioned earlier, Liberopoulos and Hu (1995) study ordering properties of optimal hedging points for some special systems. Among those, we used their Systems 1 and 2 with two and three non-deficient and one deficient states to test our simulation runs. After several runs with different parameter settings (with two non-deficient and one deficient states of both System 1 and System 2), we observed that when the load of the system is increased (by either increasing the demand or decreasing the maximum production rates), the proportion of the difference between optimal hedging points to the values of the optimal hedging points themselves decreases. That is, the difference between the optimal hedging points stays constant but because of the increased load, the optimal hedging points increase and the difference between them become negligible compared to their values. Example 5.5.4 is an experiment with their System 1 in which there are two hedging points and as the load, ρ, is increased they become close to each other.

Machine states	Demand states	Initial point A	Initial point B	Initial point C	z_A^*	z_B^*	z_C^*
4	1	5.00	10.00	20.00	19.97	18.30	18.42
4	2	5.00	10.00	20.00	21.68	21.74	22.47
4	3	5.00	10.00	20.00	34.88	29.74	29.72
4	4	5.00	10.00	20.00	29.47	30.03	28.46
5	1	5.00	10.00	20.00	20.10	21.52	21.11
5	2	5.00	10.00	20.00	20.98	22.30	21.95
5	3	5.00	10.00	20.00	29.55	28.89	28.63
5	4	5.00	10.00	20.00	28.82	31.25	33.81
7	1	5.00	10.00	20.00	7.54	7.29	7.33
7	2	5.00	10.00	20.00	9.04	9.00	9.07
7	3	5.00	10.00	20.00	27.65	22.87	21.65
7	4	5.00	10.00	20.00	23.11	21.48	21.38
8	1	5.00	10.00	20.00	7.40	7.35	7.41
8	2	5.00	10.00	20.00	9.49	9.44	10.29
8	3	5.00	10.00	20.00	19.96	21.23	20.45
8	4	5.00	10.00	20.00	23.59	21.42	20.34
9	1	5.00	10.00	20.00	8.31	8.30	8.36
9	2	5.00	10.00	20.00	10.99	12.22	11.45
10	1	5.00	10.00	20.00	13.91	13.70	13.91
10	2	5.00	10.00	20.00	16.03	14.69	13.97
10	3	5.00	10.00	20.00	28.44	28.30	28.17
10	4	5.00	10.00	20.00	27.44	28.42	26.50
$J_T(z)$		28.634	25.116	25.687	22.087	22.085	22.087
$P_{blog}(z)$		0.193072	0.123462	0.057995	0.090841	0.090899	0.090921

Initial point	N	Total CPU time
A	105	2h 22 min 49 sec
B	58	1h 18 min 47 sec
C	55	1h 15 min 53 sec

Table 5.8: Optimal solution and the run-time information for Example 5.5.3

Example 5.5.4. System 1 of Liberopoulos and Hu (1995) consist of 3 machine states. The transition rate diagram of machine states are given in Figure 5.4 and the parameter settings of the system are given in Table 5.9. The optimal hedging points of this system are: $z_1^* = 11.5$ and $z_2^* = 8.40$ when the demand rate d is 5 and the corresponding load ρ is 0.44 . When we increase the load by increasing the demand, we obtain the results summarized in Table 5.10. This leads to the following conjecture: As the load increases to one, the difference between the optimal hedging points become negligible compared to their values. In other words, the optimal hedging points appear to get close to each other as the load of the system increases.

$$\begin{bmatrix} -2 & 2 & 0 \\ 4 & -5 & 1 \\ 0 & 2 & -2 \end{bmatrix}$$

Figure 5.4: Transition rate matrix of machine states in Example 5.5.4

Machine states	1	2	3
Maximum production rate	20	40	0

c^+	c^-	$c^+/(c^+ + c^-)$
0.1	1	0.090910

Table 5.9: System specifications of Example 5.5.4

d	ρ	z_1^*	z_2^*	$z_1^* - z_2^*$	$\frac{z_1^*-z_2^*}{z_1^*}$
5	0.44	11.50	8.40	3.10	0.27
6	0.52	17.57	14.30	3.27	0.19
7	0.61	26.92	23.37	3.55	0.13
8	0.70	41.96	38.24	3.72	0.09
9	0.79	69.13	66.05	3.08	0.04
10	0.87	136.15	132.05	4.10	0.03

Table 5.10: Effect of load in Example 5.5.4

5.5.2.2 Irrelevant Hedging Point

Simulation runs of System 2 of Liberopoulos and Hu (1995) with two non-deficient and one deficient states show that when one of the non-deficient states, say State 1, has a very small long run average maximum production rate compared to the demand rate (i.e., $r_1 > d$ but $r_1\pi_1 \ll d$), the hedging point of State 1 becomes "irrelevant"; that is, the value it takes does not really matter. The optimal hedging point of non-deficient State 2 determines the optimal hedging point of the system. A similar result is obtained with systems having three non-deficient and one deficient states, where $r_1\pi_1 \ll d$, $r_2\pi_2 + r_3\pi_3 > d$, $r_2\pi_2 < d$, and $r_3\pi_3 < d$ hold but it is not the case that $r_2\pi_2 \ll d$ or $r_3\pi_3 \ll d$. In this case, z_1 becomes irrelevant and z_2 and z_3 determine the optimal hedging points of the system. This leads to the following conjecture:

Assume that there is a non-deficient state i which has a very small average maximum production rate compared to the demand rate in the long run, that is, $r_i > d$ but $r_i\pi_i \ll d$, but this is not the case for the other non-deficient states in the long run, that is, $r_j\pi_j \ll d$ does not hold for $\forall j, j \neq i$. For such a system, the optimal hedging point of state i is irrelevant and the optimal hedging points of the system are determined by the optimal hedging points of states j.

5.5.2.3 Effect of Transition Diagram and Transition Probabilities

The more the machine states communicate with each other and the more the state transition probabilities are close to uniform distribution, the more the optimal hedging points get close to each other. In the simulation runs up to systems with 10-15 non-deficient states, we observed that when the machine states form a closed set in which all states communicate with each other with a uniform transition distribution, all of the optimal hedging points are very close to each other. As the transition probabilities between the machine states become less uniform, the optimal hedging points move away from each other.

5.5.2.4 Effect of Demand and Production Variability

In the simulation runs of Example 5.5.3, we have observed that as the production and demand variability increase, the optimal hedging points also increase to hold enough safety stock against the uncertainty in production and demand rates. When the variability in production is not very high, the hedging points get more separated from each other as the demand variability increases. When production variability is high, the hedging points are first separated from each other and then get closer to each other as the demand variability increases. These observations confirm a similar comment made by Tan (2001) for a small system with two machine and two demand states.

5.5.2.5 Flatness of the Objective Function and Effect of Cost Ratio on Flatness

From the simulation runs of Example 5.5.3, we have also observed that as the ratio c^+/c^- gets smaller, that is the system has very low holding cost compared to its shortage cost, the objective function value does not change much with considerable increases in hedging points. This is very intuitive since the holding cost is very low, a large increase in the hedging level is reflected to the cost function with a relatively very small effect. Moreover, for a given ratio of c^+/c^-, the objective function is flatter near the optimum points and its flatness increases as the number of hedging points increases.

5.5.2.6 Non-convexity of the Objective Function

As also mentioned in Brémaud et al. (1997) the objective function (5.1) is generally not convex in systems with two or more non-deficient machine states. Figure 5.5(a) shows the nonconvex average cost function with respect to hedging points of machine states 3 and 4 of Example 5.5.1; whereas Figure 5.5(b) depicts its level sets. Despite this non-convexity, we reached to the same final solution for Examples 5.5.1 and 5.5.2 when we started from several different initial points. In comparison, in Example 5.5.3 when we started from different initial points, we reached to different final points which is not unexpected when dealing with nonconvex functions. As mentioned earlier, in the absence of convexity, a nonlinear optimization program like E04UCC (rather than a specialized global optimization code) can only guarantee to find a local optimum which may or may not be a global optimum. In addition, we should point out that we did not encounter any numerical difficulties in finding the local optimal solutions even in large systems despite the nonconvex nature of the objective function. We also note that at different local optimal solutions, the objective function values are very close to each other suggesting also multiple optima.

5.5.2.7 Clustering Effect on Hedging Points

It can be observed from the optimal solutions of Examples 5.5.2 and 5.5.3 (Tables 5.6 and 5.8) that the optimal hedging points of some system states are very close to each other and form clusters. We have observed this behavior in several systems that we experimented with. Unfortunately, how to correctly predict which hedging points will form clusters is not obvious from the problem data in general. It mostly depends on the topology of the transition diagrams and the transition rates of machine and demand states.

5.5.3 Optimization of Constrained Production Control Problems

In this part, we report the results of two more numerical experiments. We modified our unconstrained minimization problem (5.1) to minimization of the probabilistically constrained problem (5.12) as given below:

$$\min_{z\geq 0} \quad \lim_{T\to\infty} \frac{1}{T}\int_0^T c^+X^+(t)dt \tag{5.12}$$

$$\text{s.t.} \quad P_{blog}(z) \leq \delta.$$

How to modify (5.7) and (5.8) to account for the change of objective function is almost trivial and omitted. As mentioned earlier, the constraint function $P_{blog}(z)$ is the long run proportion of time that the system is in backlog state and it is desirable to keep it below a pre-specified constant δ. As T gets large, the long run proportion of time that the system is in backlog state can be computed as $P_{blog}(z) = B(T)/T$, where $B(T)$ is the cumulative time the system spends in backlog during T. Similar to the recursive expressions (5.7) and (5.8), for $C(t_{n+1})$ and $\partial C(t_{n+1})/\partial z_{ij}$, respectively, it is possible to write recursive expressions for $B(t_{n+1})$ and $\partial B(t_{n+1})/\partial z_{ij}$ as follows:

$$\begin{aligned} B(t_{n+1}) \quad &= \quad B(t_n) + I_{\{X(t_n)<0\}} I_{\{X(t_{n+1})<0\}} \tau_{n+1} + \\ &\quad\quad I_{\{X(t_n)\geq 0\}} I_{\{X(t_{n+1})<0\}} \frac{X(t_{n+1})}{u(t_n)} + I_{\{X(t_{n+1})\geq 0\}} I_{\{X(t_n)<0\}} \frac{-X(t_n)}{u(t_n)} \quad \text{and} \end{aligned}$$

$$\begin{aligned} \frac{\partial B(t_{n+1})}{\partial z_{ij}} \quad &= \quad \frac{\partial B(t_n)}{\partial z_{ij}} + I_{\{X(t_n)<0\}} I_{\{X(t_{n+1})<0\}} \frac{\partial \tau_{n+1}}{\partial z_{ij}} + \\ &\quad\quad \frac{1}{u(t_n)} [I_{\{X(t_n)\geq 0\}} I_{\{X(t_{n+1})<0\}} \frac{\partial X(t_{n+1})}{\partial z_{ij}} - I_{\{X(t_{n+1})\geq 0\}} I_{\{X(t_n)<0\}} \frac{\partial X(t_n)}{\partial z_{ij}}]. \end{aligned}$$

How these can be incorporated to the IPA algorithm in Appendix 5.D is trivial and omitted.

Figure 5.6(a) shows the $P_{blog}(z)$ function of the system of Example 5.5.1 which is again nonconvex, as one can clearly see from its level sets in Figure 5.6(b). Therefore, it is possible that when we start from different initial points, we may find different local optimal solutions which will indeed be the case for Examples 5.5.5 and 5.5.6 as explained below. We will explain that, in Example 5.5.5, this multiplicity is actually caused by multiple optima. Unfortunately, as in Example 5.5.6, it is not generally possible to know the source of such multiplicity without additional testing; i.e., whether it is due to several local isolated optima or multiple optima or both. Again, we also should note that despite the non-convexity of $P_{blog}(z)$, we did not encounter any numerical difficulties in our experiments in solving even larger problems.

Upon observing how flat both the objective function and the constraint around the solution points, we used 10^{-5} as $OptTol$ in Examples 5.5.5 and 5.5.6 below. In the presence of nonlinear constraints, one needs to specify an additional parameter, $ConsTol$, which is a measure of largest constraint violation that is acceptable at an optimal point; we set $ConsTol$ to 10^{-6}.

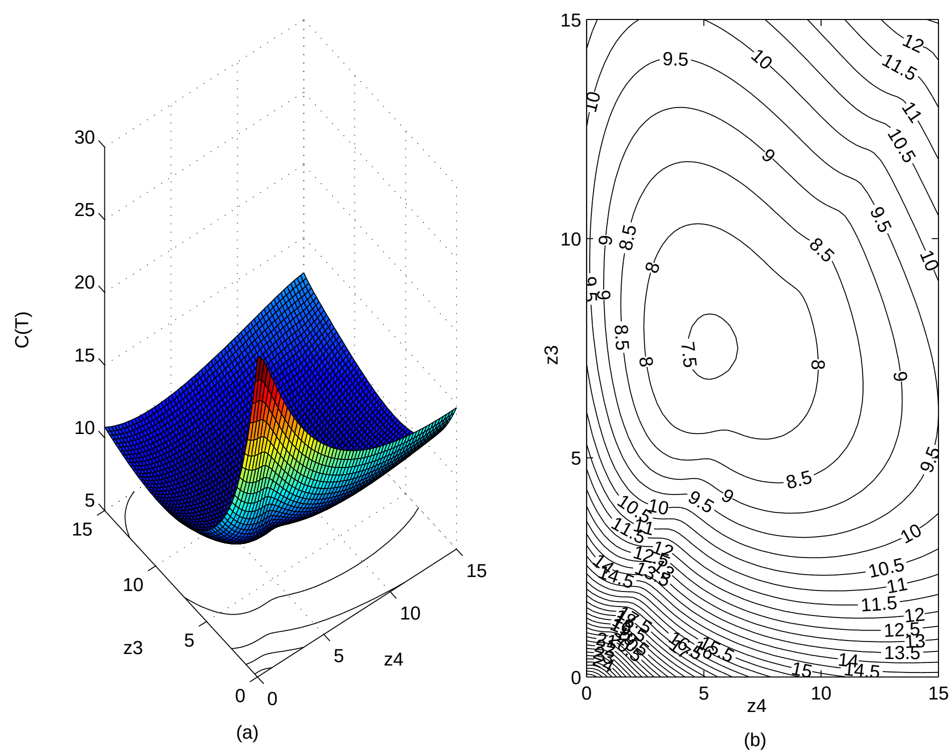


Figure 5.5: Cost function of Example 5.5.1 with respect to hedging points of machine states 3 and 4

Example 5.5.5. (Probabilistically Constrained Problem of Example 5.5.2)

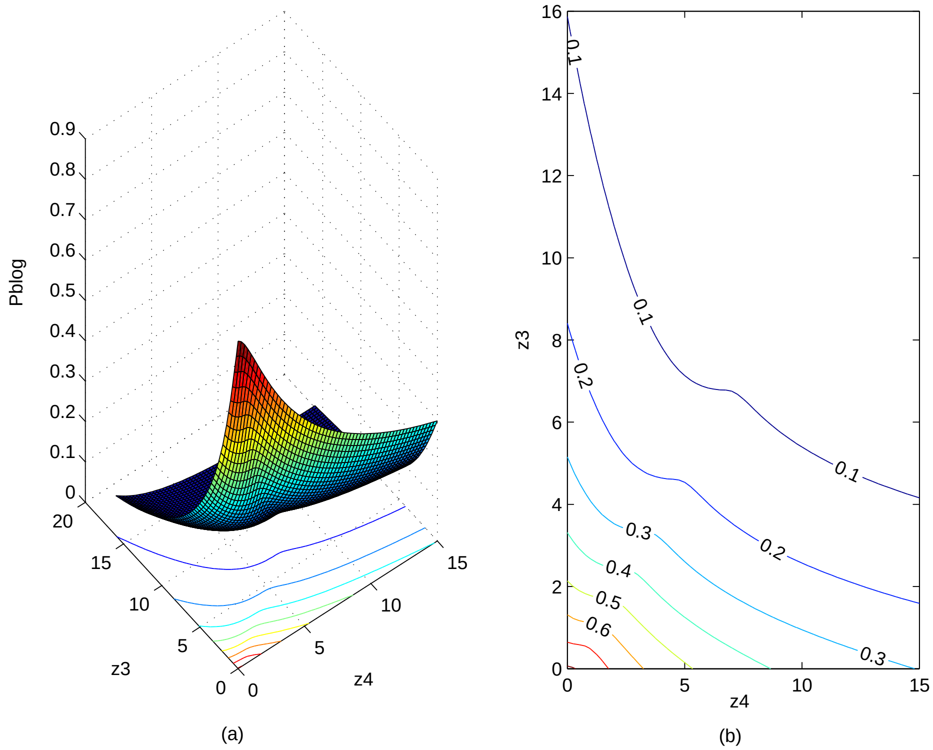


Figure 5.6: $P_{blog}(z)$ of Example 5.5.1 with respect to the hedging points of machine states 3 and 4

Here, we solved the probabilistically constrained problem for the system of Example 5.5.2 (of Section 5.5.1) with $\delta = 0.05$. Note that δ is equal to 0.090910 at the optimal hedging level for the unconstrained problem. In Table 5.11, it can be seen that when we start from different initial points (A and B), we find different optimal solutions (z_A^* and z_B^*, respectively). In Table 5.11, the more significant differences between the optimum hedging levels appear to be in z_{34}^* and z_{44}^*. In order to understand better the behavior of the objective function and the constraint, we draw both functions with respect to z_{34} and z_{44} by keeping the other hedging points at their optimal levels reached from the initial point B; see Figures 5.7(a) and 5.7(b). The contour lines of these functions illustrated in Figure 5.7(c) show that the lowest contour line of the cost function lying below $P_{blog}(z) = 0.05$ is at 13.253 and it overlaps with the contour line of $P_{blog}(z) = 0.05$ at a range of hedging points. In this range, while z_{44} is anywhere above 27, z_{34} can vary between 30 and 32 and from the performance side, all these points are equally good. We mark this range using a bold box in Figure 5.7(c). Indeed, both optimal solutions reported in Table 5.11 are located in that box, illustrating a case of multiple optima.

Example 5.5.6. (**Probabilistically Constrained Problem of Example 5.5.3**)

We also solve the probabilistically constrained problem for the system of Example 5.5.3 (of Section

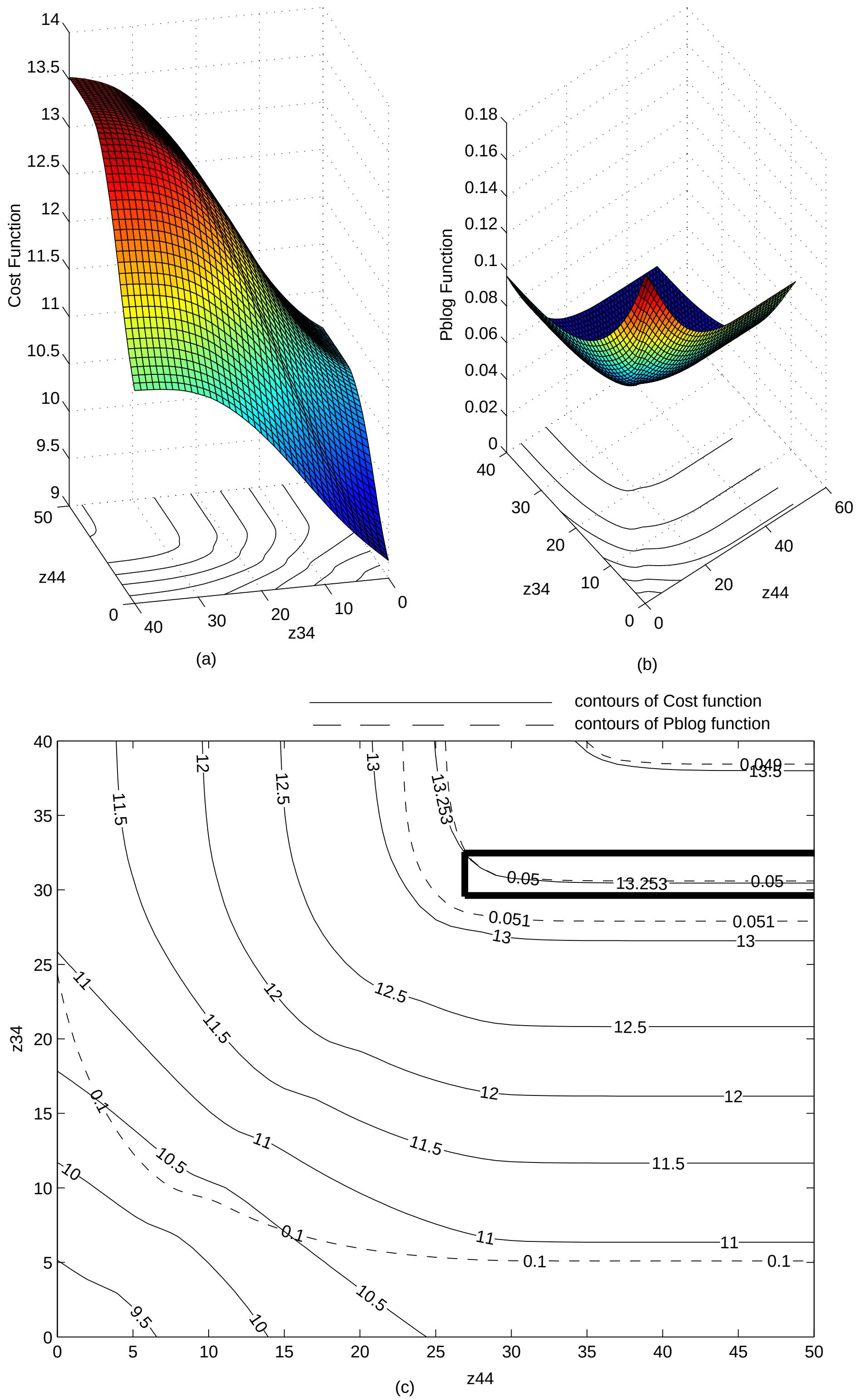


Figure 5.7: Average cost and $P_{blog}(z)$ of Example 5.5.5 with respect to z_{34} and z_{34}.

Machine states	3	3	3	3	4	4	4	4		
Demand states	1	2	3	4	1	2	3	4	$J_T(z)$	$P_{blog}(z)$
Initial point A	5.00	5.00	5.00	5.00	5.00	5.00	5.00	5.00	2.812	0.261616
Initial point B	25.00	25.00	25.00	25.00	25.00	25.00	25.00	25.00	20.596	0.036370
z_A^*	12.22	15.54	30.43	30.95	9.37	13.50	27.05	41.13	13.259	0.050000
z_B^*	12.22	15.62	30.30	31.41	9.41	13.53	27.29	28.03	13.253	0.050000

Initial point	N	Total CPU time
A	46	19 min 43 sec
B	30	12 min 59 sec

Table 5.11: Optimal solution and the run-time information for Example 5.5.5

5.5.1) with $\delta = 0.05$. The results are shown in Table 5.12. Similar to Example 5.5.5, when we start from different initial points, we find different optimal solutions with very similar performances. In the absence of other constraints and/or other selection criteria, one can choose any one of these local optimal solutions since all perform equally well.

5.6 Conclusions

We investigated the problem of finding optimal hedging points in a stochastic fluid model of a production/inventory system by simulation-based optimization. The model is fairly general in terms of the number of machine and demand states and state transitions. On the modeling and simulation side, a GSMP representation provided a general and transparent description for the evolution of the system and was crucial for obtaining sample-path derivatives. On the optimization side, an implementation of sample-path optimization enabled us to optimize systems with more than twenty hedging points and probabilistic constraints.

The empirical performance of sample-path optimization in our numerical examples is very satisfactory. For small problems, the optimal solution always coincides with previously reported solutions. For large problems, although there are no reported benchmark solutions, the solutions obtained always pass a number of consistency checks. This is very promising for other potential applications of this approach and we consider it another encouraging sign that sample-path optimization will contribute to the solution of difficult problems with several variables and/or constraints. Furthermore, from our numerical experiments, we obtain insights on the structure of optimal hedging points for much larger systems than is usually reported in the literature. In particular, we observe that production and demand variability has an important effect on the optimal threshold levels of a system. As the production and demand variability increases, the optimal threshold levels also increase to hold enough safety stock against the uncertainty in production and demand rates. Moreover, as the utilization of a system converges to one, the optimal

Machine states	Demand states	Initial point A	Initial point B	Initial point C	z_A^*	z_B^*	z_C^*
4	1	5.00	10.00	20.00	57.92	25.71	26.88
4	2	5.00	10.00	20.00	26.43	28.56	24.82
4	3	5.00	10.00	20.00	64.33	49.75	40.39
4	4	5.00	10.00	20.00	82.52	52.16	37.49
5	1	5.00	10.00	20.00	25.85	28.26	28.12
5	2	5.00	10.00	20.00	26.65	34.22	28.56
5	3	5.00	10.00	20.00	69.72	48.23	39.02
5	4	5.00	10.00	20.00	75.45	52.24	38.01
7	1	5.00	10.00	20.00	14.32	15.22	14.97
7	2	5.00	10.00	20.00	16.97	16.78	16.59
7	3	5.00	10.00	20.00	45.99	42.67	30.05
7	4	5.00	10.00	20.00	40.13	39.54	32.70
8	1	5.00	10.00	20.00	15.25	14.64	14.48
8	2	5.00	10.00	20.00	17.38	17.98	17.44
8	3	5.00	10.00	20.00	42.78	29.03	29.57
8	4	5.00	10.00	20.00	58.12	39.23	29.48
9	1	5.00	10.00	20.00	15.25	15.69	15.61
9	2	5.00	10.00	20.00	17.56	16.31	20.75
10	1	5.00	10.00	20.00	21.78	22.56	21.32
10	2	5.00	10.00	20.00	21.99	14.53	23.02
10	3	5.00	10.00	20.00	65.56	45.69	37.79
10	4	5.00	10.00	20.00	75.00	42.19	35.19
$J_T(z)$		3.590	7.816	16.958	16.655	16.615	16.586
$P_{blog}(z)$		0.193072	0.123462	0.057995	0.050000	0.050000	0.050000

Initial point	N	Total CPU time
A	79	1h 45 min 18 sec
B	50	1h 07 min 14 sec
C	87	1h 56 min 38 sec

Table 5.12: Optimal solution and the run-time information for Example 5.5.6

threshold levels get close to each other and converge to a single threshold level.

The hedging point policy in the context of manufacturing flow control is an example of a general class of policies for stochastic fluid systems where the optimal control (i.e., controllable flow rate) for each state of the external stochastic process depends on a single threshold parameter expressed as the quantity of fluid in some buffer (inventory or queue). The flow rate is then adjusted depending on this threshold. A number of other interesting and important problems, other than manufacturing flow control, fall into this framework. In production/marketing problems involving dynamic pricing, the optimal price may vary as a function of thresholds (i.e., decrease price if inventory is higher than a threshold). In subcontracting/manufacturing problems, when to subcontract depends on the current inventory (or order queue) level (i.e., use a subcontractor when the number of waiting orders exceeds a threshold). Similarly, in several admission control or routing problems in telecommunications, there is typically a threshold type policy structure (i.e., admit new arrivals of a given class if the buffer is lower than a threshold, reject otherwise). Our general approach should be applicable to such problems with certain modifications.

In addition, there are other important classes of stochastic optimization problems that are not typically modelled by stochastic fluid models but that involve the optimization of a finite number of threshold parameters in terms of certain buffer quantities. In revenue management, which fare classes to open are typically determined by (possibly time-dependent) thresholds on the number of seats/resources available. Even though these problems may require different approaches for simulation and derivative estimation, the performance of sample-path optimization in our numerical results indicates that it is a promising approach for the optimization side of such problems.

5.A Proof of Stability Condition in Eq. (5.3)

Recall that the state of the system at time t is $(\alpha(t), \beta(t))$ where $\alpha(t)$ is the state of the machine ($\alpha(t) = 1, 2, \ldots, K$) and $\beta(t)$ is the state of the demand ($\beta(t) = 1, 2, \ldots, M$), and the number of possible states of the system is KM, counting all combinations of $(\alpha(t), \beta(t))$. Tan (2002b) shows how a system with two machine states and two demand states can be represented by an equivalent system with four machine states coping with a constant demand rate. Below we generalize this basic idea for a system with KM states.

Consider a system state $k = (i, j)$ where $\alpha(t) = i, \beta(t) = j$. Let $\tilde{r}_k = r_i + \sum_{i=1}^{M} d_i - d_j$ be the maximum production rate and $p_k = \sum_{i=1}^{M} d_i - d_j$ be the minimum production rate associated with state k. Let $\tilde{d} = \sum_{i=1}^{M} d_i$ be the constant demand rate in the equivalent representation. Then we can label state k as deficient if $\tilde{r}_k < \tilde{d}$, non-deficient if $\tilde{r}_k > \tilde{d}$, and zero if $\tilde{r}_k = \tilde{d}$. This way state $k = (i, j)$ gets the same label in both representations:

$i)$ deficient: if $\tilde{r}_k < \tilde{d}$ then $r_i + \sum_{i=1}^{M} d_i - d_j < \sum_{i=1}^{M} d_i$ and $r_i < d_j$,

$ii)$ non-deficient: if $\tilde{r}_k > \tilde{d}$ then $r_i + \sum_{i=1}^{M} d_i - d_j > \sum_{i=1}^{M} d_i$ and $r_i > d_j$,

$iii)$ zero: if $\tilde{r}_k = \tilde{d}$ then $r_i + \sum_{i=1}^{M} d_i - d_j = \sum_{i=1}^{M} d_i$ and $r_i = d_j$.

Let $\tilde{v}(t)$ denote the actual production rate and $\tilde{u}(t) = \tilde{v}(t) - \tilde{d}$ denote the fill-in rate at time t in the equivalent representation; fill-in rate specifies the total rate of change in the inventory level process at time t. The control $\tilde{v}(t)$ is again completely determined by the specifications of z_k's for non-deficient and zero states as:

$$\tilde{v}(t) = \begin{cases} \tilde{r}_k & \text{if } X(t) < z_k \\ \tilde{d} & \text{if } X(t) = z_k \\ p_k & \text{if } X(t) > z_k \end{cases}$$

and for deficient states $\tilde{v}(t) = \tilde{r}_k$ regardless of the inventory position.

Then for non-deficient and zero states $\tilde{u}(t)$ becomes:

$$\tilde{u}(t) = \begin{cases} \tilde{r}_k - \tilde{d} = r_i - dj & \text{if } X(t) < z_k \\ \tilde{d} - \tilde{d} = 0 & \text{if } X(t) = z_k \\ p_k - \tilde{d} = -d_j & \text{if } X(t) > z_k \end{cases}$$

and for deficient states $\tilde{u}(t) = \tilde{r}_k - \tilde{d} = r_i - dj < 0$ which again coincide with the fill-in rate, $u(t) = v(t) - d_{\beta(t)}$, of the original system representation.

Now, let ϕ_k ($k = 1, 2, .., KM$) denote the stationary probability of being in system state $k = (i, j)$. Brémaud et al. (1997) show that a sufficient condition for stability of a system with KM machine states and a constant demand rate $\tilde{d}$ is given by:

$$\sum_{k=1}^{KM} \phi_k \tilde{r}_k > \tilde{d}. \tag{5.13}$$

The left side of (5.13) is $E[\tilde{r}]$, the expected maximum production rate of the system. $\tilde{r}_k$ is a random variable satisfying $\tilde{r}_k = r_i + \tilde{d} - d_j$. Hence, $E[\tilde{r}] = E[r] + \tilde{d} - E[d]$ with $E[r]$ being the expected maximum production rate of the machine and $E[d]$ being the expected demand. If we let π_i ($i = 1, 2, \ldots, K$) to denote the stationary probability of being in machine state i and q_j ($j = 1, 2, \ldots, M$) denote the stationary probability distribution of being in demand state j, then we can calculate $E[r]$ and $E[d]$ as follows: $E[r] = \sum_{i=1}^{K} \pi_i r_i$ and $E[d] = \sum_{j=1}^{M} q_j d_j$. After plugging these in the stability equation (5.13) and simplifying, we get:

$$\sum_{i=1}^{K} \pi_i r_i > \sum_{j=1}^{M} q_j d_j.$$

5.B Similarity Properties

The sample path (obtained by fixing $\omega \in \Omega$) of the underlying stochastic process operated at the base hedging vector z is called the *nominal path*. The sample path (with the same ω) of the same process operated at a perturbed hedging vector $z + \Delta z_{ij}$ is called the *perturbed path*. Whenever we would like to indicate the value of a quantity in the perturbed path, we will put a superscript p to the corresponding quantity in the nominal path. For example, t_n^p denotes the occurrence time of the nth event and $X^p(t)$ denotes the inventory level at time t, in the perturbed path.

Definition 5.B.1. A nominal path and a perturbed path are *similar* in $[0, t_k]$ if the order of events in the nominal path, $[0, t_k]$, is the same as the order of events in the perturbed path, $[0, t_k^p]$.

As mentioned in Section 5.4, "similarity" of a nominal path and a perturbed path is a standard issue one needs to deal with when developing IPA algorithms and we address it in detail in this appendix. In particular, we show that along any sample path of finite length, say k events, with min $\{\tau_n | n = 1, 2, \ldots, k\} > 0$, there is always a $\Delta z_{ij} > 0$ (or $\Delta z_{ij} < 0$, whose size depends on the sample path) small enough such that increasing (decreasing) z by Δz_{ij} does not cause any event order change; that is, the perturbed path and the nominal path are similar.

To this end, notice that, under our the "hit and stick" assumption, if t_n corresponds to the event $\mathscr{UH}$ or $\mathscr{DH}$, the length of time that $X(t)$ spends at z_{ij} (as a consequence of the nth event), τ_{n+1} satisfies: $\tau_{n+1} \geq \varepsilon > 0$. In addition to "hit and stick", in the arguments below, without loss of generality we assume that either T (total simulation time) or the initial conditions are chosen in a way that ensures the end of simulation event $\mathscr{T}_f$ does not coincide with any other event.

Lemma 5.B.2. *Assume that the nominal and the perturbed paths (with an initial perturbation of Δz_{ij}) are similar in $[0,t_k]$. Then for all n ($n = 1,2,\ldots,k$), the machine and demand states in the nominal path at the nth event time are equal to the machine and demand states in the perturbed path at the corresponding event time. That is $\alpha(t_n) = \alpha^p(t_n^p)$ and $\beta(t_n) = \beta^p(t_n^p)$ for all n ($n = 1,2,\ldots,k$). Furthermore, if $e^*(t_n) \in \{\mathscr{MF},\mathscr{DG}\}$, then $t_{n+1} = t_{n+1}^p$ and $e^*(t_n^p) = \mathscr{MF}$ (if $e^*(t_n) = \mathscr{MF}$) or $e^*(t_n^p) = \mathscr{DG}$ (if $e^*(t_n) = \mathscr{DG}$) for all n ($n = 1,2,\ldots,k$).*

Proof: The proof is by induction on the event instances t_n and is omitted. ∎

Lemma 5.B.3. *Assume that the nominal and the perturbed paths (with an initial perturbation of Δz_{ij}) are similar in $[0,t_k]$. Then $X^p(t) - X(t) \leq \Delta z_{ij}$, $\forall t \in [0,t_k]$. Furthermore, at all event times t_n ($n = 1,2,\ldots,k$), we have $X^p(t_n^p) - X(t_n) = \Delta z_{ij}$ or $X^p(t_n^p) - X(t_n) = 0$.*

Proof: We will only prove the second part of the lemma (the first part is an easy consequence) and will argue through induction. We start the induction at time $t_0 = 0$ with $\alpha(0) = i, \beta(0) = j$, and $X(0) = X^p(0) = x_0$. Without loss of generality, we assume that ij is a non-deficient system state. (Because in that case all possible events $\{\mathscr{MF},\mathscr{UH},\mathscr{DH},\mathscr{DG},\mathscr{T}_f\}$ may occur; whereas in zero or deficient states, only a subset of these events can take place.)

If $e^*(t_0) \in \{\mathscr{MF},\mathscr{DG},\mathscr{T}_f\}$, then $t_1 = t_1^p = \tau_1 = \tau_1^p$ and $X(t_1) = X^p(t_1^p)$. If $e^*(t_0) = \mathscr{UH}$, we should have $X(0) = X^p(0) = x_0 < z_{ij}$ and $u(t_1) = r_i - d_j$. With an initial perturbation of Δz_{ij}, we get $t_1^p = t_1 + \Delta z_{ij}/(r_i - d_j)$ and $X^p(t_1^p) = X(t_1) + \Delta z_{ij}$. Similarly, if $e^*(t_0) = \mathscr{DH}$, we should have $X(0) = X^p(0) = x_0 > z_{ij}$ and $u(t_1) = -d_j$. With Δz_{ij} as initial perturbation, we get $t_1^p = t_1 - \Delta z_{ij}/d_j$ and $X^p(t_1^p) = X(t_1) + \Delta z_{ij}$.

Now assume that the stated property holds until (and including) the event transition times t_n and t_n^p. Let $\alpha(t_n) = k$ and $\beta(t_n) = m$. By Lemma 5.B.2, we have $\alpha^p(t_n^p) = k$ and $\beta^p(t_n^p) = m$. Next observe that we only need to consider the case where $k = i$ and $m = j$, because the others will leave everything unchanged.

We first consider the case $X(t_n) = X^p(t_n^p)$. If the last event at t_{n-1} is $\mathscr{MF}$ or $\mathscr{DG}$, that is $e^*(t_{n-1}) \in \{\mathscr{MF},\mathscr{DG}\}$, then by Lemma 5.B.2, we have $t_n = t_n^p$. Hence, the induction step is the same as the beginning step. If $e^*(t_{n-1}) \in \{\mathscr{UH},\mathscr{DH}\}$, as we are in state km with $k = i$ and $m = j$, the corresponding inventory levels at t_n and t_n^p should be $X(t_n) = z_{ij}$ and $X^p(t_n^p) = z_{ij} + \Delta z_{ij}$; this is a contradiction. Hence, we can conclude that if $X(t_n) = X^p(t_n^p)$, then $e^*(t_{n-1}) \in \{\mathscr{MF},\mathscr{DG}\}$ and the induction step will be the same as the beginning step.

Next, we consider the case $X^p(t_n^p) = X(t_n) + \Delta z_{ij}$. We again need to differentiate two cases in which the last event can be either in $\{\mathscr{MF}, \mathscr{DG}\}$ or $\{\mathscr{UH}, \mathscr{DH}\}$.

If $e^*(t_{n-1}) \in \{\mathscr{MF}, \mathscr{DG}\}$, then by Lemma 5.B.2, we have $t_n = t_n^p$. Now we need to consider the possible events at t_{n+1}. If $e^*(t_n) \in \{\mathscr{MF}, \mathscr{DG}, \mathscr{T}_f\}$, then $\tau_{n+1} = \tau_{n+1}^p$. Since the system states are also the same, $X^p(t_n^p)$ and $X(t_n)$ will change equal amounts; that is $u(t_n) = u^p(t_n^p)$. Using $X(t_{n+1}) = X(t_n) + \tau_{n+1}u(t_n)$, we get $X^p(t_{n+1}^p) = X(t_{n+1}) + \Delta z_{ij}$. If $e^*(t_n) \in \{\mathscr{DH}, \mathscr{UH}\}$, as we are in state km with $k = i$ and $m = j$, the corresponding inventory levels at t_{n+1} and t_{n+1}^p should be $X(t_{n+1}) = z_{ij}$ and $X^p(t_{n+1}^p) = z_{ij} + \Delta z_{ij}$; hence the induction step holds.

If $e^*(t_{n-1}) \in \{\mathscr{DH}, \mathscr{UH}\}$, then the next event could be $\mathscr{MF}, \mathscr{DG}$, or $\mathscr{T}_f$; i.e., $e^*(t_n) \in \{\mathscr{MF}, \mathscr{DG}, \mathscr{T}_f\}$. Since the system will stay at its hedging level until the next event, and $X^p(t_n^p)$ and $X(t_n)$ will stay at the same levels with $u(t_n) = u^p(t_n^p) = 0$. Therefore, $X^p(t_{n+1}^p) = X(t_{n+1}) + \Delta z_{ij}$ and the induction step again holds. ∎

Remark 5.B.4. As a direct consequence of Lemma 5.B.3, we know that we can always select $\Delta z_{ij} > 0$ small enough such that if $X(t_n) \geq 0$ (or $X(t_n) < 0$), then $X^p(t_n^p) \geq 0$ (or $X^p(t_n^p) < 0$, respectively). This ensures that when $\Delta z_{ij} > 0$, $I_{\{X(t_n(z) \geq 0)\}} = I_{\{X(t_n(z+\Delta z_{ij}) \geq 0)\}}$ and $I_{\{X(t_n(z) < 0)\}} = I_{\{X(t_n(z+\Delta z_{ij}) < 0)\}}$ for any n.

Similarly as a consequence of Lemma 5.B.3, we can always select $\Delta z_{ij} < 0$ small enough such that if $X(t_n) > 0$ (or $X(t_n) < 0$), then $X^p(t_n^p) > 0$ (or $X^p(t_n^p) < 0$, respectively). When $X(t_n) = 0$, $e^*(t_{n-1})$ has to be either $\mathscr{DH}$ or $\mathscr{UH}$. In either case, we get $X^p(t_n^p) = 0$ as well. In order to see that $e^*(t_{n-1})$ cannot be $\mathscr{MF}, \mathscr{DG}$, or $\mathscr{T}_f$, note that $X(t_n) = 0$ means $\tau_n = |X(t_{n-1})u_{n-1}|$. This in turn would mean $|X(t_{n-1})u_{n-1}| = \operatorname{argmin}\{W^M(t_{n-1}), W^D(t_{n-1}), T - t_{n-1}\}$ whose probability is zero since the probability of a continuous random variable being equal to a specific value is always zero.

Theorem 5.B.5. $\exists\, \varepsilon_0 > 0$ *such that if any threshold* z_{ij} *is perturbed less than* ε_0 *(i.e.,* $|\Delta z_{ij}| < \varepsilon_0$*), then the nominal and the perturbed paths are similar.*

Proof: Let $\Delta z_{ij} > 0$ be the initial perturbation. We will argue inductively as follows. Let $X(t_0) = X(0) = X^p(0)$ and let the initial machine state $\alpha(0) = k$ and $\beta(0) = m$ where without loss of generality, km is assumed to be a non-deficient state. First note that, if $km \neq ij$ then $e^*(t_0) = e^*(t_0^p)$, $t_1 = t_1^p$, and $X(t_1) = X^p(t_1^p)$ trivially. Therefore, we consider the case where the perturbation is associated with the hedging point $km = ij$; that is $z_{km}^p = z_{km} + \Delta z_{km}$, with $\Delta z_{km} > 0$. By Lemma 5.*B*.2, we know that $e^*(t_0^p) = \mathscr{MF}$ if $e^*(t_0) = \mathscr{MF}$ or $e^*(t_0^p) = \mathscr{DG}$ if $e^*(t_0) = \mathscr{DG}$; hence $X(t_1) = X^p(t_1^p)$ and $t_1 = t_1^p$ trivially. The only interesting case happens if $e^*(t_0) \in \{\mathscr{UH}, \mathscr{DH}, \mathscr{T}_f\}$.

If $e^*(t_0) = \mathscr{UH}$, then this implies that $X(t_0) < z_{km}$ and $t_1 = \tau_1 = (z_{km} - X(0))/u(t_0)$. Now, the possible events in the perturbed path are on $\{\mathscr{MF}, \mathscr{DG}, \mathscr{UH}, T_f\}$. If $e^*(t_0^p) = \mathscr{MF}, \mathscr{DG}$, or T_f, then $t_1^p = \tau_1^p = W^M(t_0^p), W^D(t_0^p)$, or T, respectively. If $e^*(t_0^p) = \mathscr{UH}$, then $t_1^p = \tau_1^p = t_1 + \Delta z_{km}/u(t_0)$ as illustrated in Figure 5.8(a). We need to show that if we select Δz_{km} small enough, an event other than $\mathscr{UH}$ does not occur on the perturbed path; that is $e^*(t_0^p) = \mathscr{UH}$. This can be guaranteed if $W^M(t_0^p) > t_1 + \Delta z_{km}/u(t_0)$, $W^D(t_0^p) > t_1 + \Delta z_{km}/u(t_0)$, and $T > t_1 + \Delta z_{km}/u(t_0)$. It is obvious that we can always ensure these conditions by selecting Δz_{km} such that $\Delta z_{km} < \min\{u(t_0)(W^M(t_0^p) - t_1), u(t_0)(W^D(t_0^p) - t_1), u(t_0)(T - t_1)\}$.

If $e^*(t_0) = \mathscr{DH}$, then this implies that $X(t_0) > z_{km}$ and $t_1 = \tau_1 = (X(0) - z_{km})/ - u(t_0)$. Since $e^*(t_0) = \mathscr{DH}$, we know that $t_1 = \tau_1 < \min\{W^M(t_0), W^D(t_0), T\}$. The possible events for the perturbed path this time are $\{\mathscr{MF}, \mathscr{DG}, \mathscr{DH}$, and $T_f\}$ and $e^*(t_0^p) = \text{argmin}\{W^M(t_0^p), W^D(t_0^p), (X(0) - (z_{km} + \Delta z_{km}))/ - u(t_0), T\}$. Since $W^M(t_0^p) = W^M(t_0), W^M(t_0^p) = W^M(t_0)$, and $(X(0) - (z_{km} + \Delta z_{km}))/ - u(t_0) < t_1$ from Figure 5.8(b), we get $e^*(t_0^p) = \mathscr{DH}$.

If $e^*(t_0) = T_f$ then $t_1 = \tau_1 = T$. We should consider two cases here:

(a) In case $X(t_0) < z_{km}$ as illustrated in Figure 5.8 (a), we know that $t_1 = T < \min\{W^M(t_0), W^D(t_0), (z_{km} - X(0))/u(t_0)\}$. The possible events for the perturbed path are $\{\mathscr{MF}, \mathscr{DG}, \mathscr{UH}$, and $T_f\}$ and $e^*(t_0^p) = \text{argmin}\{W^M(t_0^p), W^D(t_0^p), ((z_{km} + \Delta z_{km}) - X(0))/u(t_0), T\}$. Since $e^*(t_0) = T_f$, it is easy to see that $T < W^M(t_0^p) = W^M(t_0), T < W^M(t_0^p) = W^M(t_0)$, and $T < ((z_{km} + \Delta z_{km}) - X(0))/u(t_0)$; therefore we get $e^*(t_0^p) = T_f$.

(b) In case $X(t_0) > z_{km}$ as illustrated in Figure 5.8 (b), we know that $t_1 = \tau_1 = T < \min\{W^M(t_0), W^D(t_0), (X(0) - z_{km})/ - u(t_0)\}$. The possible events for the perturbed path are $\{\mathscr{MF}, \mathscr{DG}, \mathscr{DH}$, and $T_f\}$ and $e^*(t_0^p) = \text{argmin}\{W^M(t_0^p), W^D(t_0^p), ((X(0) - (z_{km} + \Delta z_{km}))/(-u(t_0)), T\}$. We need to show that if we select Δz_{km} small enough, an event other than T_f does not occur on the perturbed path; that is $e^*(t_0^p) = T_f$. This can be guaranteed if $T < W^M(t_0^p)$, $T < W^D(t_0^p)$, and $T < ((X(0) - (z_{km} + \Delta z_{km}))/(-u(t_0))$. We know that $T < W^M(t_0^p) = W^M(t_0), T < W^M(t_0^p) = W^M(t_0)$ since $e^*(t_0) = T_f$. We can also ensure that $T < ((X(0) - (z_{km} + \Delta z_{km}))/(-u(t_0))$ holds by selecting Δz_{km} small enough ($0 < \Delta z_{km} < (X(0) - z_{km} + u(t_0)T)$). This proves the first step of the induction.

We now assume that the nominal path and the perturbed path are similar until (and including) the event times t_n and t_n^p. Let $\alpha(t_n) = k$ and $\beta(t_n) = m$. By Lemma 5.B.2, we have $\alpha^p(t_n^p) = k$ and $\beta^p(t_n^p) = m$. We have to show that the next events on both the nominal and perturbed paths are identical; that is $e^*(t_n) = e^*(t_n^p)$.

By the argument in Lemma 5.B.3, we know that the similarity of the paths until the nth event implies that $X^p(t_n^p) = X(t_n)$ or $X^p(t_n^p) = X(t_n) + \Delta z_{ij}$. First assume that, $X^p(t_n^p) = X(t_n)$. If $ij = km$, then we know from the proof of Lemma 5.B.3 that $e^*(t_{n-1}) = e^*(t_{n-1}^p) \in \{\mathscr{MF}, \mathscr{DG}\}$ which implies that $t_n = t_n^p$. Also if $ij \neq km$ and if $e^*(t_{n-1}) = e^*(t_{n-1}^p) \in \{\mathscr{MF}, \mathscr{DG}\}$, then $t_n = t_n^p$. In both cases, $X^p(t_n^p) = X(t_n)$ and $t_n = t_n^p$; hence the rest of the argument is identical to the first step of the induction and the next events are identical. We now focus on the case where $ij \neq km$ but $e^*(t_{n-1}) = e^*(t_{n-1}^p) \in \{\mathscr{UH}, \mathscr{DH}\}$. This implies that $e^*(t_n)$ and $e^*(t_n^p)$ can only be a machine or demand state transition or end of simulation, on both nominal and perturbed paths; hence $e^*(t_n) = e^*(t_n^p)$.

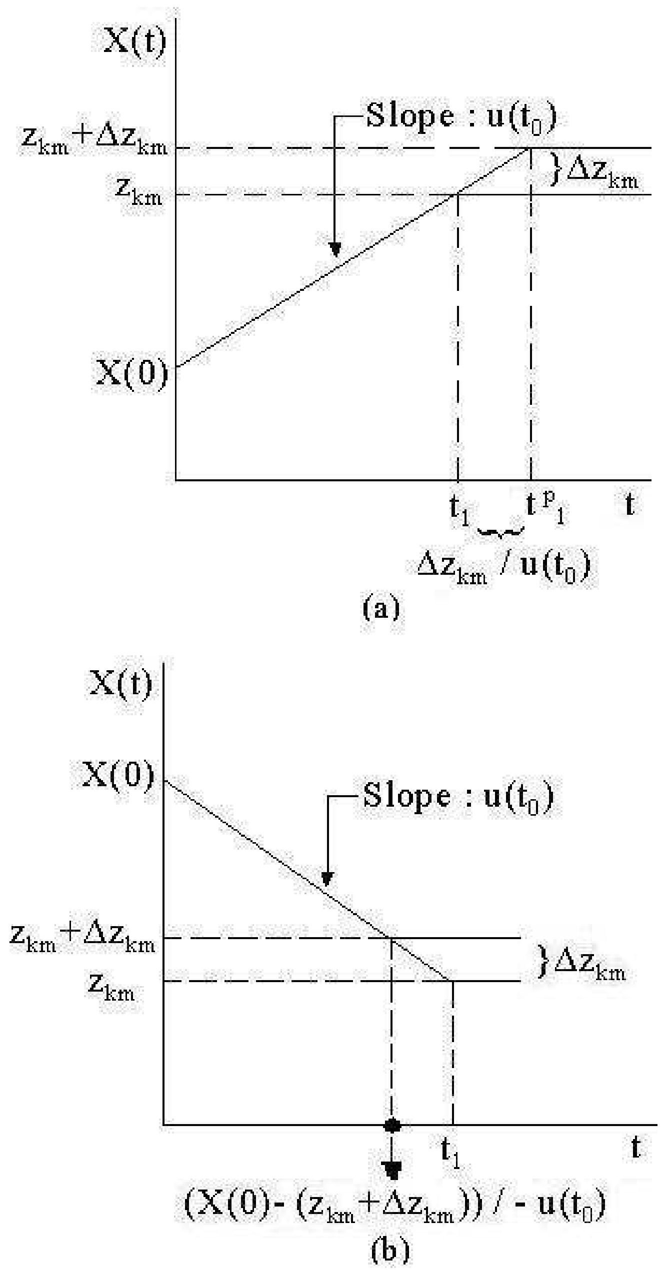


Figure 5.8: Similarity of nominal and perturbed paths in $[0, t_1]$

Next assume that, $X^p(t_n^p) = X(t_n) + \Delta z_{ij}$. Here, we again need to consider two cases $ij = km$ and $ij \neq km$ separately.

In case $ij \neq km$, $e^*(t_{n-1}) = e^*(t_{n-1}^p)$ is in $\{\mathscr{MF}, \mathscr{DG}\}$. Because if $e^*(t_{n-1}) = e^*(t_{n-1}^p) \in \{\mathscr{UH}, \mathscr{DH}\}$ then $X(t_n) = X^p(t_n^p) = z_{km}$ and by Lemma 5.B.2 we have $t_n^p = t_n$. By Lemma 5.B.2, again $e^*(t_n^p) = \mathscr{MF}$ if $e^*(t_n) = \mathscr{MF}$ and $e^*(t_n^p) = \mathscr{DG}$ if $e^*(t_n) = \mathscr{DG}$. If $e^*(t_n) = \mathscr{UH}$ (i.e., $X(t_n) < z_{km}$), then $\tau_{n+1} = (z_{km} - X(t_n))/u(t_n)$. The possible events for the perturbed path are in

$\{\mathscr{MF}, \mathscr{DG}, \mathscr{UH}, \mathscr{T}_f\}$ and we can always select Δz_{ij} small enough such that $X^p(t_n^p) = X(t_n) + \Delta z_{ij} < z_{km}$. Since $e^*(t_n) = \mathscr{UH}$, we know that $\tau_{n+1} < \min\{W^M(t_n), W^D(t_n), T - t_n\}$. We also know that $e^*(t_n^p) = \operatorname{argmin}\{W^M(t_n^p), W^D(t_n^p), (z_{km} - X^p(t_n^p))/u(t_n), T - t_n^p\}$. Since $W^M(t_n) = W^M(t_n^p), W^D(t_n) = W^D(t_n^p), T - t_n = T - t_n^p$ and $(z_{km} - X^p(t_n^p))/u(t_n) < \tau_{n+1}$ as illustrated in Figure 5.9(a), we get $e^*(t_n^p) = \mathscr{UH}$. If $e^*(t_n) = \mathscr{DH}$ (i.e., $X(t_n) > z_{km}$), then $\tau_{n+1} = (X(t_n) - z_{km})/(-u(t_n))$. The possible events for the perturbed path are in $\{\mathscr{MF}, \mathscr{DG}, \mathscr{DH}, \mathscr{T}_f\}$ ($\mathscr{UH}$ is excluded since $X^p(t_n^p) > X(t_n) > z_{km}$). If $e^*(t_n^p) = \mathscr{MF}, \mathscr{DG}$, or T_f, then $\tau_{n+1}^p = W^M(t_n^p), W^D(t_n^p)$, or $T - t_n^p$, respectively. If $e^*(t_n^p) = \mathscr{DH}$ then $\tau_{n+1}^p = (X^p(t_n^p) - z_{km})/(-u(t_n))$ which implies that $\tau_{n+1}^p = \tau_{n+1} + \Delta z_{ij}/(-u(t_n))$ as illustrated in Figure 5.9(b). We need to show that if we select Δz_{ij} infinitesimally small enough, an event other than $\mathscr{DH}$ does not occur on the perturbed path, that is $e^*(t_n^p) = \mathscr{DH}$. This will be ensured if $W^M(t_n^p) > \tau_{n+1} + \Delta z_{ij}/(-u(t_n)), W^M(t_n^p) > \tau_{n+1} + \Delta z_{ij}/(-u(t_n))$, and $T - t_n^p > \tau_{n+1} + \Delta z_{ij}/(-u(t_n))$. It is obvious that we can always ensure these conditions by selecting Δz_{ij} such that $\Delta z_{ij} < \min\{-(W^M(t_n^p) - \tau_{n+1})u(t_n), -(W^D(t_n^p) - \tau_{n+1})u(t_n), -(T - t_n^p - \tau_{n+1})u(t_n)\}$.

In case $ij = km$, we have $X^p(t_n^p) = X(t_n) + \Delta z_{km}$. First consider the case where $e^*(t_{n-1}) = e^*(t_{n-1}^p) \in \{\mathscr{MF}, \mathscr{DG}\}$; then we have $t_n^p = t_n$ by Lemma 5.B.2. By Lemma 5.B.2 again, $e^*(t_n^p) = \mathscr{MF}$ if $e^*(t_n) = \mathscr{MF}$ or $e^*(t_n^p) = \mathscr{DG}$ if $e^*(t_n) = \mathscr{DG}$. If $e^*(t_n) = \mathscr{UH}$ (i.e., $X(t_n) < z_{km}$), then $\tau_{n+1} = (z_{km} - X(t_n))/u(t_n)$. The possible events for the perturbed path are in $\{\mathscr{MF}, \mathscr{DG}, \mathscr{UH}, \mathscr{T}_f\}$ ($\mathscr{DH}$ is excluded since $X^p(t_n^p) = X(t_n) + \Delta z_{km} < z_{km} + \Delta z_{km}$). Since $e^*(t_n) = \mathscr{UH}$, we know that $\tau_{n+1} < \min\{W^M(t_n), W^D(t_n), T - t_n\}$. We also know that $e^*(t_n^p) = \operatorname{argmin}\{W^M(t_n^p), W^D(t_n^p), ((z_{km} + \Delta z_{km}) - X^p(t_n^p))/u(t_n), T - t_n^p\}$. Since $W^M(t_n) = W^M(t_n^p)$, $W^D(t_n) = W^D(t_n^p), T - t_n = T - t_n^p$, and $((z_{km} + \Delta z_{km}) - X^p(t_n^p))/u(t_n) = \tau_{n+1}$, we get $e^*(t_n^p) = \mathscr{UH}$. The argument is similar for the case $e^*(t_n) = \mathscr{DH}$ with $\tau_{n+1} = (X^p(t_n^p) - (z_{km} + \Delta z_{km}))/(-u(t_n))$. We now consider the case where $e^*(t_{n-1}) = e^*(t_{n-1}^p) \in \{\mathscr{UH}, \mathscr{DH}\}$. This implies that $e^*(t_n)$ and $e^*(t_n^p)$ can only be a machine or demand state transition or end of simulation, on both nominal and perturbed paths; hence $e^*(t_n) = e^*(t_n^p)$. This completes the induction argument.

The case with $\Delta z_{ij} < 0$ can be argued similarly. ▮

5.C Proofs of Lemmas in Section 5.4 for Gradient Estimation

Proof of Lemma 5.4.2. Observe that at any event time t_n, $r(e_{n+1}, t_n)$ is equal to either one of 0, 1, $r_i - d_j$, or $-d_j$ for some i and j. Thus $r(e_{n+1}, t_n)$ is a function of r_i and d_j for some i, j and t_n. Of those variables only t_n depends on the hedging points. As a consequence of similarity for small enough Δz, $t_n(z + \Delta z y_{ij})$ is still the occurrence time of the nth event. Therefore we must have $r(e_{n+1}, t_n(z + \Delta z y_{ij})) = r(e_{n+1}, t_n(z))$ and

$$\frac{\partial r(e_{n+1}, t_n)}{\partial z_{ij}} = \lim_{\Delta z \to 0} \frac{r(e_{n+1}, t_n(z + \Delta z y_{ij})) - r(e_{n+1}, t_n(z))}{\Delta z} = 0. \blacksquare$$

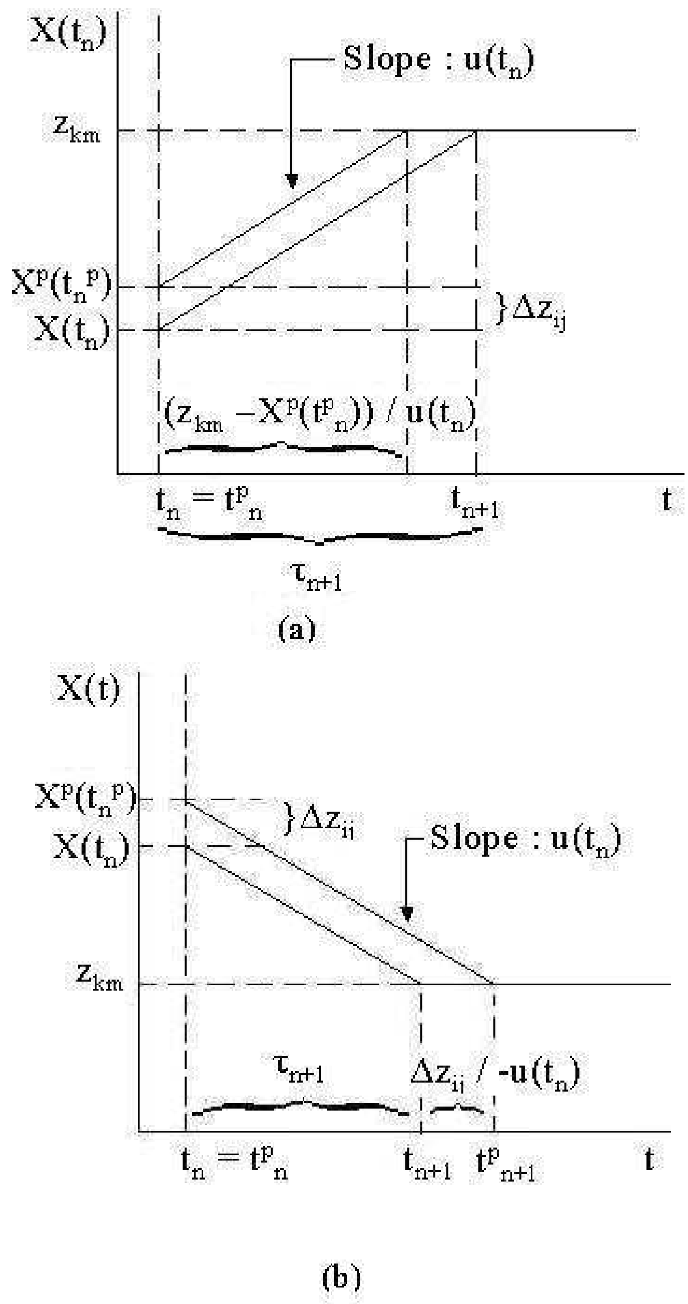


Figure 5.9: Similarity of nominal and perturbed paths in $[t_n, t_{n+1}]$

Proof of Lemma 5.4.3. Similar to Lemma 5.4.2. ∎

Proof of Lemma 5.4.4. Since $u(t)$ is constant between adjacent events, we have $X(t_n) = X(t_{n-1}) + u(t_{n-1})\tau_n$. From Lemma 5.4.3, we know that $\partial u(t_{n-1})/\partial z_{ij}$ is 0; hence the result follows. ∎

Proof of Lemma 5.4.5. Recall that $W^M(t_n) = W^M(t_{n-1}) - \tau_n$ and $W^D(t_n) = W^D(t_{n-1}) - \tau_n$. The result follows by taking the partial derivatives of these equations at z. ∎

Proof of Lemma 5.4.6. Recall that

$$k(e_{n+1}, t_n) = \begin{cases} W^M(t_n) & \text{if } e_{n+1} = \mathscr{MF}, \\ z_{\alpha(t_n)\beta(t_n)} - X(t_n) & \text{if } e_{n+1} = \mathscr{UH}, \\ X(t_n) - z_{\alpha(t_n)\beta(t_n)} & \text{if } e_{n+1} = \mathscr{DH}, \\ W^D(t_n) & \text{if } e_{n+1} = \mathscr{DG}, \\ T - t_n & \text{if } e_{n+1} = \mathscr{T}_f. \end{cases}$$

The result follows immediately. ∎

Proof of Lemma 5.4.7. Direct consequence of (5.6) and Lemma 5.4.2. ∎

Proof of Lemma 5.4.8. Since we start at time $t_0 = 0$, we have $X(t_0) = x_0, W^M(t_0) = W^M, W^D(t_0) = W^D$, and $C(t_0) = 0$. Clearly these are all independent of z_{ij}. Since $t_1 = \tau_1 = k(e_1, t_0)/r(e_1, t_0)$, it is enough to show the following facts about $k(e_1, t_0)$ and $r(e_1, t_0)$: (a) $r(e_1, t_0) \neq 0$; (b) $r(e_1, t_0)$ has partial derivatives at z for all z and $i = 1, \dots, K$, $j = 1, \dots, M$, which are finite; (c) $k(e_1, t_0)$ has partial derivatives at z for all z and $i = 1, \dots, K$, $j = 1, \dots, M$, which are finite. Since $e_1 \in E(t_0)$, (a) is immediate from the definition of $E(t)$ and (b) follows from Lemma 5.4.2. To see (c), observe that $E(t_0) = \{\mathscr{MF}, \mathscr{UH}, \mathscr{DG}, \mathscr{T}_f\}$ or $E(t_0) = \{\mathscr{MF}, \mathscr{DH}, \mathscr{DG}, \mathscr{T}_f\}$. Hence we have

$$k(e_1, t_0) = \begin{cases} W^M & \text{if } e_1 = \mathscr{MF}, \\ z_{\alpha(t_0)\beta(t_0)} - x_0 & \text{if } e_1 = \mathscr{UH}, \\ x_0 - z_{\alpha(t_0)\beta(t_0)} & \text{if } e_1 = \mathscr{DH}, \\ W^D & \text{if } e_1 = \mathscr{DG}, \\ T - t_0 & \text{if } e_1 = \mathscr{T}_f. \end{cases}$$

Thus $k(e_1, t_0)$ is a differentiable function of z_{ij} for $i = 1, \dots, K$, $j = 1, \dots, M$ whose partial derivative is $-1, 0$, or 1. ∎

Proof of Lemma 5.4.9. From Lemma 5.B.3 in Appendix 5.B, we have $X(\omega, t_n(z + \Delta z_{ij})) - X(\omega, t_n(z)) \leq \Delta z_{ij}$. Then $|C(\omega, t_n(z + \Delta z_{ij})) - C(\omega, t_n(z))| \leq t_n |\Delta z_{ij}|(c^+ + c^-)$. Hence $|C(\omega, T(z + \Delta z_{ij})) - C(\omega, T(z))| \leq T|\Delta z_{ij}|(c^+ + c^-)$ and $|J_T(\omega, z + \Delta z_{ij}) - J_T(\omega, z)| \leq |\Delta z_{ij}|(c^+ + c^-)$. Therefore

$$\left|\frac{J_T(\omega, z + \Delta z_{ij}) - J_T(\omega, z)}{\Delta z_{ij}}\right| \leq (c^+ + c^-). \blacksquare$$

5.D Pseudo Code for Simulation and IPA Algorithm

5.D.1 Variables and Procedures

Let w_i^M be the mean holding time in machine state i and w_j^D be the mean holding time in demand state j. We use sample(μ) to denote a sample taken from a prescribed distribution with mean μ.
The following events are defined for the GSMP representation when the current state is $\alpha\beta$:

$\mathscr{M}\mathscr{F}$: the end of the sojourn time of the current machine state
$\mathscr{U}\mathscr{H}$: reaching the hedging point of current system state from below
$\mathscr{D}\mathscr{H}$: reaching the hedging point of current system state from above
$\mathscr{D}\mathscr{G}$: the end of the sojourn time of the current demand state
$\mathscr{T}_f$: end of simulation.

The following notation and procedures are used in the simulation/gradient estimation algorithm:

INF	: A very large constant
t	: Current time
τ	: Time to occurrence of triggering event
M	: Number of demand states
K	: Number of machine states
α	: Current machine state
β	: Current demand state
d_j	: Rate of the demand state j
z_{ij}	: Hedging point of state ij
r_i	: Maximum production rate of machine state i
v	: Current actual production rate of the system
u	: Current fill-in-rate of the system
X	: Current inventory level of the system
X_{prev}	: Previous inventory level of the system
$cost$	: Current cumulative cost of the system
Avg_cost	: Average cost of the system
No_Events	: Number of possible events
$E[i]$	: Possible Event i, $i = 1, 2, \dots, No\_Events$
$T[i]$	: Time to occurrence of $E[i]$
K^*	: Index of triggering event
W^M	: Time to occurrence of event $\mathscr{M}\mathscr{F}$
W^D	: Time to occurrence of event $\mathscr{D}\mathscr{G}$

T_{remain}	: Time to occurrence of event $\mathscr{T}_f$
t_{sim}	: Total simulation time
x_0	: Initial inventory level
α_0	: Initial machine state
β_0	: Initial demand state
$Csum[i][j]$	: Accumulator of $\partial C/\partial z_{ij}$
$Tsum[i][j]$	: Accumulator of $\partial t_n/\partial z_{ij}$
$Xsum[i][j]$	: Accumulator of $\partial X(t_n)/\partial z_{ij}$
$XPrevsum[i][j]$	: Accumulator of $\partial X(t_{n-1})/\partial z_{ij}$
$WMsum[i][j]$	: Accumulator of $\partial W^M(t_n)/\partial z_{ij}$
$WDsum[i][j]$	: Accumulator of $\partial W^D(t_n)/\partial z_{ij}$
$Temp[i][j]$	: Accumulator of $\partial \tau_n/\partial z_{ij}$
Gen_Sys	: Boolean variable which is 1 if it is a general system ($M > 1$), 0 otherwise
$S[i][j]$	: State of the system at machine state i and demand state j.

$$I_{\{\mathscr{A}\}} = \begin{cases} 1 & \text{if } \mathscr{A} \text{ holds} \\ 0 & \text{otherwise.} \end{cases}$$

Procedure: Determine_Event_Space()

$E[0] = \mathscr{MF}$; $E[1] = \mathscr{UH}$; $E[2] = \mathscr{DH}$;

IF ($Gen\_Sys == 1$) THEN $\{E[3] = \mathscr{DG}$; $E[4] = \mathscr{T}_f$; $No\_Events = 4$;$\}$

ELSE $\{E[3] = \mathscr{T}_f$; $No\_Events = 3$;$\}$

Procedure: Determine_States()

FOR i=1 TO K DO

FOR j=1 TO M DO

$\{$IF ($r_i < d_j$) THEN $S[i][j] =$'deficient';
ELSE IF ($r_i > d_j$) THEN $S[i][j] =$'non-deficient';
ELSE IF ($r_i == d_j$) THEN $S[i][j] =$'zero';$\}$

Procedure: Set_Uv()

IF ($S[\alpha][\beta] ==$'non-deficient') THEN

IF ($X < z_{\alpha\beta}$) THEN $v = r_\alpha$;
ELSE IF ($X == z_{\alpha\beta}$) THEN $v = d_\beta$;
ELSE IF ($X > z_{\alpha\beta}$) THEN $v = 0$;

ELSE IF ($S[\alpha][\beta] ==$'zero') THEN

IF ($X > z_{\alpha\beta}$) THEN $v = 0$; ELSE $v = r_\alpha$;

ELSE IF($S[\alpha][\beta]$ ==‘deficient’) THEN $v = r_\alpha$;

$u = v - d_\beta$;

5.D.2 Simulation Code

0. Initialization:

$t = 0$; $\tau = 0$; $X = x_0$; $X_{prev} = 0$; $T_{remain} = t_{sim}$; $cost = 0$;
IF ($M > 1$) THEN $Gen\_Sys = 1$; ELSE $Gen\_Sys = 0$;
Determine_Event_Space();
Determine_States();
Choose initial machine state $\alpha_0 : \alpha = \alpha_0$; Choose initial demand state $\beta_0 : \beta = \beta_0$;
Set_Uv();
Generate $W^M \sim$ sample(w^M_α)
IF ($Gen\_Sys == 1$) THEN Generate $W^D \sim$ sample(w^D_β)

0(IPA). Initialization:

FOR i=1 TO K DO
FOR j=1 TO M DO

{$Csum[i][j] = 0$; $Tsum[i][j] = 0$; $Xsum[i][j] = 0$; $XPrevsum[i][j] = 0$; $WMsum[i][j] = 0$;
$Temp[i][j] = 0$;
IF ($Gen\_Sys == 1$) THEN $WDsum[i][j] = 0$;}

1. Next Local Event:

$T[0] = W^M$;
IF ($S[\alpha][\beta]$ ==‘non-deficient’) THEN

{IF ($X < z_{\alpha\beta}$) THEN { $T[1] = \frac{z_{\alpha\beta}-X}{u}$; $T[2] = INF$; }
ELSE IF ($X == z_{\alpha\beta}$) THEN { $T[1] = INF$; $T[2] = INF$; }
ELSE IF ($X > z_{\alpha\beta}$) THEN { $T[1] = INF$; $T[2] = \frac{X-z_{\alpha\beta}}{-u}$; }}

ELSE IF ($S[\alpha][\beta]$ ==‘zero’) THEN

{$T[1] = INF$;
IF ($X > z_{\alpha\beta}$) THEN $T[2] = \frac{X-z_{\alpha\beta}}{-u}$; ELSE $T[2] = INF$ }

ELSE IF($S[\alpha][\beta]$ ==‘deficient’) THEN

$\{T[1] = INF; T[2] = INF\}$

IF ($Gen\_Sys == 1$) THEN $\{T[3] = W^D; T[4] = T_{remain};\}$ ELSE $T[3] = T_{remain}$;

2. Next Global Event:
K^* =argmin($T[i] : i = 1, 2, \ldots, No\_Events$);
$\tau = T[K^*]$; $e^* = E[K^*]$;

3A(IPA). Perturbation Generation:
Case:

$e^* = \mathscr{MF}$: FOR i=1 TO K DO
FOR j=1 TO M DO
$Temp[i][j] = WMsum[i][j]$;
$e^* = \mathscr{UH}$:FOR i=1 TO K DO
FOR j=1 TO M DO
$Temp[i][j] = \frac{1}{u}(I_{\{i=\alpha, j=\beta\}} - XPrevsum[i][j])$;
$e^* = \mathscr{DH}$:FOR i=1 TO K DO
FOR j=1 TO M DO
$Temp[i][j] = \frac{1}{-u}(-I_{\{i=\alpha, j=\beta\}} + XPrevsum[i][j])$;
$e^* = \mathscr{DG}$:FOR i=1 TO K DO
FOR j=1 TO M DO
$Temp[i][j] = WDsum[i][j]$;
$e^* = \mathscr{T}_f$: FOR i=1 TO K DO
$Temp[i][j] = -Tsum[i][j]$;

3B(IPA). Perturbation Update:
FOR i=1 TO K DO
FOR j=1 TO M DO

$\{Tsum[i][j] = Tsum[i][j] + Temp[i][j]$;
$Xsum[i][j] = XPrevsum[i][j] + u \times Temp[i][j]$;
IF $e^* = \mathscr{MF}$ THEN $WMsum[i][j] = 0$; ELSE $WMsum[i][j] = WMsum[i][j] - Temp[i][j]$;
IF $e^* = \mathscr{DG}$ THEN $WDsum[i][j] = 0$; ELSE $WDsum[i][j] = WDsum[i][j] - Temp[i][j];\}$

4. Update:

$t = t + \tau$; $X_{prev} = X$; $X = X + u\tau$;

$$\begin{aligned} cost &= cost + \\ & I_{\{X_{prev}\geq 0\}} I_{\{X\geq 0\}} c^{+}[X\tau - \frac{u\tau}{2}\tau] + \\ & I_{\{X_{prev}<0\}} I_{\{X<0\}} c^{-}[-X\tau + \frac{u\tau}{2}\tau] + \\ & I_{\{X_{prev}\geq 0\}} I_{\{X<0\}} \{c^{+}[\frac{-X_{prev}}{2}\frac{X_{prev}}{u}] + c^{-}[\frac{-X}{2}\frac{X}{u}]\} + \\ & I_{\{X\geq 0\}} I_{\{X_{prev}<0\}} \{c^{+}[\frac{X}{2}\frac{X}{u}] + c^{-}[\frac{X_{prev}}{2}\frac{X_{prev}}{u}]\} \end{aligned}$$

4A(IPA). Update Cost Perturbation:

FOR i=1 TO K DO

FOR j=1 TO M DO

$$\begin{aligned} \{Csum[i][j] &= Csum[i][j] + \\ & I_{\{X_{prev}\geq 0\}} I_{\{X\geq 0\}} c^{+}[Xsum[i][j]\tau + XTemp[i][j] - u\tau Temp[i][j]] + \\ & I_{\{X_{prev}<0\}} I_{\{X<0\}} c^{-}[-Xsum[i][j]\tau - XTemp[i][j] + u\tau Temp[i][j]] + \\ & I_{\{X_{prev}\geq 0\}} I_{\{X<0\}} \{c^{+}[\frac{-X_{prev}XPrevsum[i][j]}{u}] + c^{-}[\frac{-Xsum[i][j]X}{u}]\} + \\ & I_{\{X\geq 0\}} I_{\{X_{prev}<0\}} \{c^{+}[\frac{Xsum[i][j]X}{u}] + c^{-}[\frac{X_{prev}XPrevsum[i][j]}{u}]\}\} \end{aligned}$$

Case:

$e^{*} = \mathscr{MF}$: Generate new machine state k; $\alpha = k$; Generate $W^{M} \sim$ sample(w^{M}_{α});

$e^{*} = \mathscr{DG}$: Generate new demand state l; $\beta = l$; Generate $W^{D} \sim$ sample(w^{D}_{β});

IF ($e^{*} \neq \mathscr{MF}$) THEN $W^{M} = W^{M} - \tau$;

IF ($Gen\_Sys == 1$) THEN

IF ($e^{*} \neq \mathscr{DG}$) THEN $W^{D} = W^{D} - \tau$;

IF ($e^{*} = \mathscr{T}_{f}$) THEN GO TO Output;

ELSE IF ($e^{*} \neq \mathscr{T}_{f}$) THEN

$\{T_{remain} = T_{remain} - \tau$; Set_Uv(); GO TO Next Local Event;$\}$

5. Output:

$Avg\_cost = cost/t_{sim}$;

5(IPA). Gradient Output:

FOR i=1 TO K DO

FOR j=1 TO M DO

$\partial C/\partial z_{ij} = Csum[i][j]/t_{sim};$

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Summary

The word *stochastic* is from a Greek origin and conveys the idea of “randomnes”. It is used to indicate that a particular subject or system contains random phenomenon that obeys probabilistic, rather than deterministic, laws. The research in this thesis deals with modelling and solving some interesting stochastic problems with contributions in two areas; namely in electricity markets and in production and inventory control. To this aim, we utilize the theory and application of optimization and equilibrium problems and simulation techniques in operations research. Chapter 2 provides the reader with the necessary background for the theory, methods, and mathematical notation used in the succeeding chapters.

One of the most interesting and recent applications in economics include electricity markets which have experienced significant changes towards deregulation and competition with the aim of improving economic efficiency. Since the introduction of competition in electricity markets by decentralization of electricity generation, many interesting research questions have been addressed in the intersection of operations research and noncooperative game theory in economics. Most of these questions originate from the challenges faced by regulators and policy makers due to the specific characteristics of electricity markets. Chapters 3 and 4 deal with the problem of generation capacity investments in deregulated electricity markets, which has gained a lot of attention since the concern about incentives to invest in adequate generation capacity was raised in the literature as well as among policy makers.

In Chapter 3, we consider the problem of resource adequacy in competitive electricity markets. Under the old-style vertically integrated monopoly structure of electricity markets before deregulation, prices were regulated and set to cover total costs and hence enable the utility to finance investment for sufficient generation capacity to meet consumers’ current and projected electricity demand. Since deregulation, the power generation sector has operated in a competitive environment and the investments in power generation capacity is no longer done by a central planner and are left to private companies responding to current and expected market prices. Thus, long term planning of power generation capacity may suffer from uncertainty and imperfections in the short-term (e.g., day ahead and real time) operation of electricity markets. Initially, when liberalisation was introduced, policy makers held the view that energy-only electricity markets would provide sufficient incentives for investment in new generation capacity. However, experience with these energy-only markets have led policy makers to change their views and raise the question whether these markets would provide sufficient incentives for investment or whether

additional policy measures would be needed to ensure these investments and thereby security of supply in competitive electricity markets. Parallel to the policy debate, many authors in the scientific literature argue that important conditions for adequate generation capacity investments in energy-only electricity markets are prices which are allowed to rise to the point where consumers would prefer to be shut off instead of paying this price (i.e., the value of lost load (VOLL)) and sufficient flexibility in demand to react to high prices. However, these conditions are not necessarily met in real energy-only electricity markets. Due to the lack-of demand-side response (e.g., smart meters) for most of the consumers and low and volatile spot market prices, reflecting short-run marginal costs and constrained by low price caps, current energy-only electricity markets may not provide incentives for investments in adequate generation capacity.

Additional market mechanisms have been discussed in the literature to remedy the resource adequacy problem in energy-only electricity markets. Chapter 3 provides analysis of three examples of these market mechanisms under perfect competition: an energy-only electricity market with VOLL pricing or a price cap, an electricity market with an additional forward capacity market, and an electricity market with operating-reserve pricing. We elaborate on both how these market mechanisms can be represented by mathematical models and the assessment of equilibrium for generation capacity investments under different market designs when future spot market conditions are not known in advance (i.e., under demand uncertainty). Under each market design, the investment decisions of generators are formulated as a two-stage equilibrium problem where generation capacities are installed in the first stage and generation takes place in future spot market at the second stage. We emphasize the link between optimization and equilibrium problems and show that, regarding the problem of generation capacity investments in perfectly competitive electricity markets, the equilibrium problems of most of these market mechanisms can be cast as optimization problems (e.g., (non)linear programs, two-stage stochastic programs). In particular, this result indicates the prevalence of optimization problems, which are fast and efficient to solve, for providing solutions to stochastic equilibrium models of real world systems with uncertainty. This result also suggests that an equilibrium of a single-stage model (so called *open loop model*) in which investment and operation decisions are made simultaneously coincides with an equilibrium of a two-stage model (so called *closed loop model*) where investment and operation decisions are made sequentially.

Two-stage equilibrium models with stochastic elements (e.g., demand uncertainty) may prove computationally challenging to solve due to the resulting large scale problems. We also utilize simulation-based methods and provide detailed algorithmic approaches for numerically tackling all the models of these different market designs. Our computational methods allow solving large-scale problems with several uncertain parameters by utilizing deterministic off-the-shelf solvers. We also illustrate that these algorithmic approaches help further reduce the computational time. Through numerical experiments, we gain insights on the impact of demand uncertainty and to what extend these different mechanisms can remedy the resource adequacy issue. In particular, we show that uncertainty of demand leads to higher total generation capacity investments and a broader mix of technologies compared to the investment decisions assuming average demand levels. Furthermore for the same VOLL level, energy-only markets with VOLL pricing tend to lead to total generation capacity below the peak load with a certain probability whereas energy markets with a forward capacity market or operating-reserve pricing result in higher investments, assuming random demand with finite support and no forced outages. Finally, the regulator decisions (e.g., reserve capacity target) in capacity markets and operating-reserve pricing can be chosen in such a way that results in very similar investment levels and fuel mix of generation capacities in both market designs.

In Chapter 4, we again consider the problem of generation capacity investments in energy-only electricity markets but this time by strategic generators. Thus, we have a closed loop model where strategic electricity generators anticipate perfectly competitive spot market outcomes with demand uncertainty while choosing their capacities and their power generation is dispatched after the level of demand is realized. In this case, an equilibrium of the closed loop model does not coincide with an equilibrium of the less complicated open loop model. In general, such closed loop models can be formulated as an equilibrium problem with equilibrium constraints (EPEC) and examples have been posed in the literature that have multiple or no equilibria. As a consequence, an equilibrium for the corresponding two-stage game may not exist or, if exists, it is in general not unique. Therefore, it is of interest to define general sets of conditions under which solutions exist and are unique, which would enhance the value of such models for policy and market intelligence purposes. We consider various types of such a closed loop model regarding the underlying price-demand relations (elastic and inelastic demand), the assumed demand uncertainty with a broad class of continuous distributions, and any finite number of generators with symmetric or asymmetric costs. We establish sufficient conditions for the random demand's probability distribution which guarantee existence and uniqueness of equilibria in most of the cases of this closed loop model. We identify a broad class of commonly used continuous probability distributions satisfying these conditions. Establishing a set of general conditions for existence and uniqueness is desirable for policy and market models including strategic firms so that users can have confidence in their solutions before solving such complicated closed loop models. Furthermore, we also show that whether strategic firms sell their power via central auction or bilateral contracts in the competitive spot market, the corresponding closed loop games yield the same equilibrium, if it exists. Thus, when the spot market is perfectly competitive, different structures of buying and selling electricity do not affect the generation capacity choices of strategic firms.

The final chapter of the thesis, Chapter 5, addresses a different stochastic problem which falls in the field of production and inventory control. A number of important problems discussed in the literature under production and inventory control involve optimization of multiple target inventory levels or hedging points in manufacturing systems. In such systems, the production and demand rates change according to some stochastic process. According to the previous studies in the literature, derivation of the closed-form analytical solution and finding the optimal hedging points are difficult for large scale production systems with several production and demand states. Chapter 5 addresses the problem of finding the optimal hedging points in such a stochastic manufacturing flow control system whose dynamics can be modelled using generalized semi-Markov processes (GSMP). The GSMP framework enables us to compute several performance measures and their sensitivities from a single simulation run for a general system with several states and fairly general state transitions. We then use a simulation-based optimization method, *sample-path optimization*, for finding optimal hedging points. We report numerical results for systems with more than twenty hedging points and service-level type probabilistic constraints. In these numerical studies, our method performs quite well even on problems with nonconvex objective functions and probabilistic constraints, which are considered very difficult to solve by current standards. In addition, we obtain insights on the structure of optimal hedging points for much larger systems than is usually reported in the literature. In particular, we observe that production and demand variability has an important effect on the the optimal hedging point levels of a system. As the production and demand variability increases, the optimal hedging point levels also increase to hold enough safety stock against the uncertainty in production and demand rates. Furthermore, as the utilization of a system converges to one, the optimal hedging point levels get close to each other and converge to a single hedging point level. Although the main focus of Chapter 5 is the analysis of a particular manufacturing flow control problem, the scope of application for the developed techniques is potentially much wider; e.g., production/marketing problems involving dynamic pricing, using capacity options and subcontracting strategies in manufacturing problems, or admission control or routing problems in telecommunications.

Finally, a key difficulty in decision making under uncertainty is in dealing with an uncertainty space that is huge and frequently leads to very large-scale stochastic models. Thus, development of alternative methods to find a solution to stochastic problems is an active area of research. Among these, we utilize a class of simulation-based methods, so called *sample-path methods* (also known as *sample average approximation methods*), to solve the stochastic problems presented in Chapters 3 and 5. In sample-path methods, a complex stochastic system is observed along a fixed sample path by using the method of common random numbers from simulation literature. This way the underlying stochastic problem is converted to a deterministic problem which provides an approximate solution to the solution of the stochastic problem. The resulting deterministic problem is then solved by fast and effective deterministic solution methods available and its solution is taken as an approximate solution of the original stochastic problem. By using sample-path methods in Chapters 3 and 5, we can solve large-scale problems with several uncertain parameters by utilizing deterministic off-the-shelf solvers.